Exactly Marginal Operators and Duality
in
Four Dimensional N=1 Supersymmetric Gauge Theory

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We show that manifolds of fixed points, which are generated by exactly marginal operators, are common in N=1 supersymmetric gauge theory. We present a unified and simple prescription for identifying these operators, using tools similar to those employed in two-dimensional N=2 supersymmetry. In particular we rely on the work of Shifman and Vainshtein relating the β-function of the gauge coupling to the anomalous dimensions of the matter fields. Finite N=1 models, which have marginal operators at zero coupling, are easily identified using our approach. The method can also be employed to find manifolds of fixed points which do not include the free theory; these are seen in certain models with product gauge groups and in many non-renormalizable effective theories. For a number of our models, S-duality may have interesting implications. Using the fact that relevant perturbations often cause one manifold of fixed points to flow to another, we propose a specific mechanism through which the N=1 duality discovered by Seiberg could be associated with the duality of finite N=2 models.

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I. INTRODUCTION

The study of conformal field theory in two dimensions has an extensive history. Among the features common in these theories are extended manifolds of fixed points, often called fixed lines, fixed planes, *etc.* These manifolds are generated by "exactly marginal" operators; at a fixed point with an exactly marginal operator $\mathcal{O}$, the addition of the operator to the Lagrangian $\delta\mathcal{L} = h\mathcal{O}$ leads to a new fixed point for a continuous range of h. To prove $\mathcal{O}$ is exactly marginal is generally very difficult; marginality at $h = 0$ is insufficient, since the dimension of $\mathcal{O}$ may vary with h. In two dimensions one has large classes of soluble models where the dimensions of operators are exactly known and fixed lines can be proven to exist. However, there are also models with N=2 supersymmetry, which, although not always soluble, have enough symmetry that it is possible to prove that an operator is exactly marginal [1,2]. As we will show, this is also true in four-dimensional N=1 supersymmetry.

The existence of marginal operators in N=1 supersymmetry is implicit in the work of various authors in the context of perturbatively finite models [3]– [7]. (For finite models, Refs. [4,5] reach much the same conclusions as we do but use more complex techniques to arrive at them; the approach of Piguet and Sibold and collaborators [5] is related to ours, though they use a very different language.) An example has also been identified in an interacting theory [9]. We will present a unified and simplified description of these phenomena, and will display many new examples, demonstrating that marginal operators are to be found throughout supersymmetric gauge theory. Many of these generate manifolds of fixed points with properties worthy of further investigation. In particular we will see several models in which the phenomenon of strong-weak coupling duality may be studied, and we will see a suggestive relation between N=2 duality and the N=1 duality studied by Seiberg [8].

Our method, which we will explain more thoroughly below, can be easily summarized. We use certain properties of $d = 4$, N=1 supersymmetric gauge theories which are similar to those of $d = 2$, N=2 supersymmetric theories [2]. Holomorphy of the superpotential implies severe restrictions; in particular, couplings of chiral fields in the superpotential are not per-turbatively renormalized [10,11]. Non-perturbative renormalizations of the superpotential are restricted by holomorphy [11,12,13]. Still, any physical coupling *is* renormalized, and its running can be expressed in terms of its canonical dimension and the anomalous dimensions of the fields that it couples. That is, corresponding to the superpotential $W = h\phi_1 \ldots \phi_n$ there is a β-function

$$\beta_h \equiv \frac{\partial h(\mu)}{\partial \ln \mu} = h(\mu)\Big(- d_W + \sum_k \Big[d(\phi_k) + \frac{1}{2}\gamma(\phi_k)\Big]\Big) \equiv h(\mu)A_h \tag{1}$$

where d_W is the canonical dimension of the superpotential, $d(\phi_k)$ is the canonical dimension of the field ϕ_k and $\gamma(\phi_k)$ is its anomalous mass dimension. We will refer to A_h as a scaling coefficient; it is related to the physical dimension of the operator $\phi_1 \ldots \phi_n$. The Wilsonian gauge couplings undergo renormalization only at the one-loop level; the physical running gauge coupling then has an exact β-function [14]

$$\beta_g \equiv \frac{\partial g(\mu)}{\partial \ln \mu} = -f(g[\mu])\Big(\Big[3C_2(G) - \sum_k T(R_k)\Big] + \sum_k T(R_k)\gamma(\phi_k)\Big) \equiv f(g[\mu])A_g \tag{2}$$

where $C_2(G)$ is the quadratic Casimir of the adjoint representation, $T(R_k)$ is the quadratic Casimir of the representation in which ϕ_k appears, and $f(g)$ is a function of the gauge coupling which may have a pole at large g but is otherwise smooth and positive. The derivation of these statements (which we will review below) requires only the scale dependence of the Wilsonian effective action and the chiral (Konishi) anomaly.

We now examine the conditions for a fixed point. If we have n independent couplings g_i in the theory, we have n β-functions $\beta_i(g_1, \ldots, g_n)$. At a fixed point, the β-functions must all vanish. This requirement imposes n conditions on the n couplings. If these conditions are independent, we expect at most isolated solutions. However, it may happen that some of the β-functions are linearly dependent. If $p < n$ is the number of linearly independent β-functions, then $\beta_i = 0$ imposes only p conditions on the n couplings, and we expect the generic solution to be an $n - p$ dimensional submanifold in the space of couplings. Of course, in no case are we guaranteed that any solutions can be found; we can only say that if a solution exists, the solution space will generically have dimension $n - p$. Translation within an extended space of fixed points corresponds to varying a marginal coupling constant. Each such coupling constant is associated with a marginal operator in the theory which remains exactly marginal within the manifold of fixed points.

As an example of the power of these observations, we review a typical two-dimensional Landau-Ginsburg model [2]. The model has n identical chiral superfields ϕ_k and a superpotential $W = \lambda \sum_k (\phi_k)^n$. We may take λ to be real and positive by redefining the fields. The symmetries require that all fields have the same anomalous dimension γ. Using (1), $d_W = 1$ and $d_k = 0$, one finds a non-trivial β-function $\beta_\lambda \propto -1 + \frac{1}{2}n\gamma(\lambda)$. If a superconformal fixed point exists, then we have $\gamma(\lambda_*) = \frac{2}{n}$. For this fixed point to be stable there must exist an $\epsilon > 0$ such that for $0 < \delta < \epsilon$, $\gamma(\lambda_* - \delta) < \frac{2}{n} < \gamma(\lambda_* + \delta)$.

Now let us add the operator $\delta W = h\phi_1\phi_2 \ldots \phi_n$ to the theory. This operator is marginal at the conjectured fixed point, and it preserves the flavor symmetry, so the anomalous dimensions of the fields remain equal to one another. The coupling h has β-function $\beta_h \propto -1 + \frac{1}{2}\sum_k \gamma(\lambda, h) \propto \beta_\lambda$, so from the two β-functions we have only one condition on the couplings λ, h. We therefore expect a complex curve of fixed points, specified by $\gamma(\lambda, h) = \frac{2}{n}$, which passes through the point $(\lambda, h) = \lambda_*, 0$ but extends out into the (λ, h) plane. This conclusion is unavoidable as long as $\gamma(\lambda, h)$ is a continuous function, as can be seen from the stability condition for the fixed point at $h = 0$.

We now turn to four-dimensional supersymmetric gauge theory. Using (1) and (2) we have searched for models which have coupling constants whose β-functions are linearly dependent. We will present a number of models in Sec. IV, many of which have appeared in the literature, which have manifolds of fixed points intersecting the point where the gauge coupling vanishes; such theories can be analyzed at weak coupling. Many (if not all) of these N=1 models, including both vector-like and chiral theories, are finite, at least in the sense in which N=4 supersymmetric theories are finite. That is, the anomalous dimensions of all chiral operators vanish, so that the effective action has no ultraviolet divergences; there can still be divergent non-chiral operator renormalizations. There is an extensive literature on finite N=1 quantum field theories in four dimensions [3]– [7], and detailed lists of these models are given in [6].

We then turn to manifolds of non-perturbative fixed points. We have found a class of theories that contains both weakly and strongly coupled models. The study of this set of ex-

amples (Sec. V) gives us confidence to apply our techniques outside the perturbative regime. We then display a number of other cases in Sec. VI, including models with chiral matter content, which have the potential for fixed manifolds at strong coupling. In this situation the candidate marginal operator is perturbatively non-renormalizable; consequently our approach is on unstable ground in these models and our conclusions are conjectural. Still, we will argue that our methods apply to effective field theories with a supersymmetric cutoff, subject to certain conditions, so that renormalizability is not a requirement.

Even if our methods do apply, they do not rule out the possibility in most interacting models that there are no fixed points anywhere. However, certain $SU(N)$ supersymmetric gauge theories with matter in the fundamental representation were recently studied by Seiberg [8], who conjectured the existence of a wide class of interacting fixed points using a variety of arguments. (The existence of interacting fixed points for large gauge groups can be verified perturbatively.) The case of $SO(N)$ and $Sp(N)$ gauge groups was outlined in [8] and the former has been fully explored in [9]. As we will show, in certain cases these fixed points have exactly marginal operators, and so we are reasonably confident that fixed lines exist in these models. The fixed points studied by Seiberg also have the special property [8] that they have at least two descriptions involving different gauge groups with different representation content. This "N=1 duality" is useful to us in certain cases where it gives additional evidence that the formulas (1) and (2) are non-perturbatively valid.

Finally we will address the issue of renormalization group flow from one manifold of fixed points to another. We will present a wide variety of examples in Sec. VII. We observe that it is possible that dualities present in certain theories are partially preserved under the renormalization group flow. In particular we will present evidence in Sec. VIII for a connection between duality of N=2 finite models and the N=1 duality observed in [8].

II. VANISHING OF SCALING ANOMALIES AND ANOMALOUS DIMENSIONS

The conditions (1) and (2) follow from simple considerations about scaling invariance in four-dimensional supersymmetric gauge theory. This was first shown by Shifman and Vainshtein [14]; we now review their argument, slightly adapted for our purposes. We will follow its presentation with a discussion of its significance and limitations.

A. Conditions for fixed points and marginal operators

Consider a non-Abelian gauge theory with n chiral fields ϕ_i and a superpotential $W(\phi_i)$. Classically the theory is scale invariant except for terms in the superpotential; the derivative of the supercurrent multiplet (which includes the energy-momentum tensor, the supercurrent, and the chiral R current [15]) gives

$$\overline{D}^{\dot{\alpha}} J_{\alpha\dot{\alpha}}\big|_{\text{classical}} = \frac{1}{3} D_\alpha \left(3W - \sum_{i=1}^{n} \phi_i \frac{\partial W}{\partial \phi_i} \right) \tag{3}$$

Quantum mechanically there are additional terms from the scaling anomalies [16] associated with the gauge one-loop β-function and the anomalous dimensions of the matter fields. These

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anomalies cause additional dependence of the supercurrent operator on the Wilsonian cutoff μ.

$$\overline{D}^{\dot\alpha} J_{\alpha\dot\alpha} = \overline{D}^{\dot\alpha} J_{\alpha\dot\alpha}\Big|_{\text{classical}} - \frac{1}{3}D_\alpha\left(\frac{b_0}{32\pi^2}W_\beta W^\beta + \frac{\overline{D^2}}{8}\sum_{i=1}^{n}\gamma_i Z_i \phi_i^\dagger e^V \phi_i\right) \qquad (4)$$

where W_β is the gauge field strength superfield, $b_0 = 3C_2(G) - \sum_i T(R_i)$ with notation as in (2), $Z_i(\mu)$ is the wave function renormalization of the field ϕ_i and $\gamma_i = -\partial \ln Z_i/\partial \ln\mu$ is its anomalous mass dimension. We have assumed here that the fields ϕ_i do not mix under renormalization. If they do mix, then a field redefinition should be performed so that the matrix of anomalous dimensions is diagonal, following which this derivation will apply.

The equation of motion for each field ϕ_i, multiplied by ϕ_i and corrected to account for the chiral (Konishi) gauge anomalies [16,17] of the theory, is

$$\frac{\overline{D}^2}{4}Z_i\phi_i^\dagger e^V \phi_i = \frac{1}{16\pi^2}T(R_i)W_\beta W^\beta + \phi_i\frac{\partial W}{\partial \phi_i} \ . \qquad (5)$$

Substituting this into the supercurrent anomaly, we find

$$\overline{D}^{\dot\alpha} J_{\alpha\dot\alpha} = \frac{1}{3}D_\alpha\left[-\frac{W_\beta W^\beta}{32\pi^2}\left(b_0 + \sum_i T(R_i)\gamma_i\right) + \left(3W - \sum_{i=1}^{n}\phi_i\frac{\partial W}{\partial \phi_i}(1 + \frac{1}{2}\gamma_i)\right)\right] \qquad (6)$$

Assuming the superpotential has the form of a polynomial

$$W(\phi_i) = \sum_s h_s W^{(s)}(\phi_i) \qquad (7)$$

where $W^{(s)}$ is a product of d_s fields, we may rewrite the anomaly as

$$\overline{D}^{\dot\alpha} J_{\alpha\dot\alpha} = -\frac{1}{3}D_\alpha\left[\frac{W_\beta W^\beta}{32\pi^2}\left(b_0 + \sum_i T(R_i)\gamma_i\right) + \sum_s h_s\left((d_s - 3)W^{(s)} + \sum_i \frac{1}{2}\gamma_i\phi_i\frac{\partial W^{(s)}}{\partial \phi_i}\right)\right] \qquad (8)$$

Thus, to have a theory with no scale dependence, the scaling coefficients

$$A_g = -\left[3C_2(G) - \sum_i T(R_i) + \sum_i T(R_i)\gamma_i\right] \qquad (9)$$

and, for each s,

$$A_{h_s} = (d_s - 3) + \frac{1}{2}\sum_i \gamma_i\frac{\partial \ln W^{(s)}}{\partial \ln \phi_i} \qquad (10)$$

must vanish. (The expression $\frac{\partial \ln W^{(s)}}{\partial \ln \phi_i}$ simply counts the number of times ϕ_i appears in $W^{(s)}$.) Near a fixed point the β-functions for the gauge coupling g and the superpotential couplings h_s must be proportional to these conditions. In our conventions, as can be read off from the classical and one-loop behavior of (8), a coupling is driven to zero if its scaling coefficient is positive and away from zero if it is negative.

The condition for a fixed point is that all scaling coefficients (9), (10) vanish. This puts n constraints on the n couplings $g, \{h_s\}$. If these constraints are linearly independent, then we expect their solutions to be isolated points in the space of coupling constants. But if only p constraints are linearly independent, then the generic solution to the vanishing of the scaling coefficients will be an $n - p$ dimensional manifold of fixed points. Of course, it is always possible that the constraints have no solutions, either because they put contradictory conditions on the anomalous dimensions, or because there are no values of the coupling constants for which the anomalous dimensions satisfy them.

B. Discussion

The exact validity of the formulas (9) and (10), even non-perturbatively, is of the utmost importance for us in this paper. Certainly this will be reliable for those models with lines of fixed points passing through weak coupling. Non-perturbative renormalizations of the superpotential are generally, by holomorphy, ultraviolet finite, and as long as we are at moderately weak coupling there should be no strange behavior in the Kähler potential which would invalidate the derivation. For non-renormalizable effective theories, however, there are potential pitfalls. In Eqs. (4) and (5) we implicitly assumed that the operators we wrote were the most relevant ones. For this to be appropriate with a non-renormalizable superpotential, we should have a substantial gauge coupling at the ultraviolet cutoff (which of course should not be taken to infinity) so that the superpotential is nearly marginal there. This can only happen outside the perturbative regime. It is therefore possible for non-renormalizable operators present in the effective Kähler potential to become marginal or relevant before the superpotential does. In this case Eqs. (4) and (5) do not properly characterize the theory. The appearance of marginal or relevant non-chiral operators in the action often signals a breakdown of the description of the theory in terms of the fields ϕ_i, as occurs in [8] for gauge group $SU(N_c)$ with $N_f \leq \frac{3}{2}N_c$ flavors in the fundamental representation. We will assume that (9) and (10) are valid in all cases in which the description in terms of the original fields still makes sense. The consistency of our results with other results in the literature [1]- [9], [20] suggest that this is so.

At a superconformal fixed point there is an R symmetry which is part of the superconformal multiplet. The multiplet contains the generator of dilations, and as a result the R charge and the dimension $D_k = d_k + \frac{1}{2}\gamma_k$ of a gauge invariant chiral superfield field are related [18,19]. In four dimensions this relation is $D = \frac{3}{2}R$, or $\gamma_k = 3R_k - 2d_k$. The conditions (9) and (10) ensure that the R charge is conserved by the superpotential and has no gauge anomaly. This is natural, since the derivative of the supercurrent contains both the scaling and R anomalies as components. One may rephrase our arguments for the existence of a marginal operator by searching for a perturbation of a known fixed point under which the R symmetry is unambiguously preserved, up to an anomaly-free $U(1)$. This approach avoids the problems of the derivation presented above, in that no assumptions are made concerning the Kähler potential. Consequently, we believe that the existence of a stable fixed point which has a unitary description in terms of the ϕ_i probably implies that the derivation of (8) does not suffer, in the vicinity of that point, from problems associated with marginal and relevant non-chiral operators. Furthermore, it is often useful to understand the physics away from but near the fixed point, and the Shifman-Vainshtein argument gives

more physical insight into the renormalization group flow than does a discussion limited to the R symmetries at a fixed point.

The operators of the theory form a representation of the superconformal algebra. This representation must be unitary. As shown in [19] and employed in [8], this puts restrictions on the dimensions of operators. Specifically, the dimension of any gauge invariant operator must either be zero (in which case the operator is the identity), one (in which case the operator is a free field), or greater than one. This will cause us to discard certain candidate marginal operators because the theory containing them would have to be non-unitary. It also ensures that renormalizable models with no gauge invariances cannot have non-trivial fixed points. Consider a Wess-Zumino theory of chiral superfields with a superpotential made from cubic couplings. By unitarity the superfields must all have dimension one or greater; for the cubic superpotential to be dimension three, all must have dimension one. But such fields must be free at a superconformal fixed point, so the superpotential flows either to zero or to strong coupling where the description in terms of the original fields breaks down. (In perturbation theory the anomalous dimensions and the β-functions are positive, so the theory is free in the infrared.) Theories of chiral superfields with non-renormalizable superpotentials can be rejected as well. Similarly, in an Abelian gauge theory, the fact that the gauge field strength superfield $W_\alpha = -\frac{1}{4}\overline{D}^2 D_\alpha V$ is gauge invariant and has dimension $\frac{3}{2}$ implies that either the theory is free in the infrared (as one expects perturbatively) or the theory flows to a region of strong coupling where the description in terms of the original fields breaks down. We will therefore consider only non-Abelian gauge theory for the remainder of the paper.

The equations (9) and (10) are special when $b_0 = 0$ and $d_s - 3 = 0$; in this case the scaling coefficients are homogeneous in the anomalous dimensions of the fields. The theory with zero gauge coupling and no superpotential, for which all anomalous dimensions vanish, is then a stable fixed point with a marginal operator. In some cases the vanishing of the scale anomaly forces the anomalous dimensions of all fields to vanish, implying the theory has a manifold of fixed points where its effective action is finite. However, in other cases our formalism does not imply this; the dimensions of some chiral operators, or, equivalently, the charges of certain fields under the R symmetry in the supercurrent, are undetermined. Despite this it is straightforward to show the anomalous dimensions always vanish at leading-loop order when Eqs. (9)-(10) are satisfied. At this time we are unable to show that these theories are finite to all orders, nor are we able to verify the claims of Kazakov to this effect [4]. Perhaps finiteness follows from the fact that these superconformal theories can be continuously deformed to zero coupling. The resolution of this issue does not affect the bulk of our results.

III. THE FIXED LINES OF N=4 AND N=2 THEORIES

The best known models with fixed lines are those of N=4 supersymmetry. These can be thought of as N=1 theories with a gauge coupling constant g and three chiral superfields in the adjoint representation Σ_i, coupled through the superpotential $W = h\Sigma_1\Sigma_2\Sigma_3$. By symmetry, the three fields have the same anomalous dimension γ. The N=4 supersymmetry requires $h = g$; but let us relax this condition. The vanishing of the scaling coefficients

$$A_g = -3C_2(G)\gamma \propto A_h = \frac{3}{2}\gamma \tag{11}$$

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puts only *one* condition on *two* couplings, namely $\gamma(g,h) = 0$. We know this is true at $g = h = 0$, so a curve of fixed points may pass through the free theory.

That such a curve exists, at least for weak coupling, can be easily confirmed from basic physical intuition or by simple calculation. For $g \gg h$ the superpotential is negligible and the theory is an gauge theory which is known to be infrared free; from (9) this corresponds to the statement that $\gamma(g \gg h)$ is negative. For $g \ll h$ the theory is a pure scalar field theory which is infrared trivial and has a Landau pole, which from (10) tells us that $\gamma(g \ll h)$ is positive. If $\gamma(g,h)$ is continuous, these two regions must be separated by a curve where γ vanishes. The one-loop formula $\gamma(g,h) = A(h^2 - g^2)$ where $A > 0$ confirms this. A similar argument applies for every weakly coupled model that we will present.

Furthermore, the behavior of $\gamma(g,h)$ on the space of couplings shows that the fixed line is infrared stable; near but off the fixed line, the sign of $\gamma(g,h)$ is such that the theory is driven to the fixed line in the infrared. The fine-tuning of the couplings which is needed to set $\gamma(g,h) = 0$ and put the theory on the fixed line is thus a natural one. (A similar situation will be found in all of the models that we study.) Another way to say this is that if the N=4 supersymmetry is broken at the ultraviolet cutoff, it will return as an accidental symmetry in the infrared. In Fig. 1 we illustrate these points.

Finally, we emphasize the simplicity of our arguments. To show that the curve of fixed points lies on the line $g = h$ would require the use of the full N=4 supersymmetry. However, only N=1 supersymmetry was used in proving the *existence* of the fixed line. Additionally, the *finiteness* of the model, which follows from $A_g \propto A_h \propto \gamma(g,h) = 0$, was derived using N=1 supersymmetry alone.

We can see a similar feature in models with N=2 supersymmetry. Consider a theory with N_f hypermultiplets in some representation R. Treated as an N=1 model [], the matter content is an adjoint representation Σ, associated with the gauge fields, and N_f hypermultiplets consisting of pairs $Q^f, \tilde{Q}_f$ in conjugate representations $R, \overline{R}$. The superpotential of the model, $W = hQ^f \Sigma \tilde{Q}_f$, preserves the flavor symmetry, so we know that all the $Q^f, \tilde{Q}_f$ have the same anomalous dimension γ_Q. The scaling coefficients are

$$A_g = -(2C_2(G) - N_f T(R)) - C_2(G)\gamma_\Sigma - N_f T(R)\gamma_Q$$
$$A_h = \tfrac{1}{2}(\gamma_\Sigma + 2\gamma_Q) \tag{12}$$

These are proportional if $b_0 = 2C_2(G) - N_f T(R) = 0$; thus, if the one-loop gauge β-function vanishes, we expect a fixed curve with $g \approx h$. (A concrete example of the above is the $SU(2)$ model with 4 hypermultiplets in the fundamental representation, which was discussed in Ref. [20].) Again this can be verified perturbatively for small coupling. Notice that our methods do not obviously prove finiteness here, since only $\gamma_\Sigma + 2\gamma_Q$ need vanish. However, by N=2 supersymmetry, the fermion in the superfield Σ must have the same dimension as the gluino W_α, whose R charge is fixed to be one and whose dimension is therefore its canonical value of $3/2$. This ensures that the dimension of Σ is its canonical value, so γ_Σ is zero and thus γ_Q vanishes as well.

IV. THEORIES WITH WEAKLY COUPLED FIXED LINES IN N=1 SUPERSYMMETRY

We now turn to N=1 supersymmetry. We begin by discussing models which have fixed lines passing through weak coupling, many of which have appeared in the literature [3]-[7]. In [4] and [5] these theories are proven to have marginal operators to all orders in perturbation theory, though the methods used are quite different from ours. We believe that our approach is simpler and gives a clearer insight into the reason for the existence of these models. Also, to the best of our knowledge, many of our comments on these examples are original.

As in the finite examples studied above, these theories must have a gauge coupling with vanishing one-loop β-function, so that the origin is a stable fixed point. The couplings in the superpotential must therefore be cubic in order that they be marginal at this point. Holomorphy and dimensional analysis [11] ensure that these dimensionless couplings are not renormalized by non-perturbative effects. Furthermore, in many cases, all anomalous dimensions of the matter fields necessarily vanish, making the effective action of these theories finite. (In cases where the anomalous dimensions are not constrained to vanish by our methods, the finiteness of the model is as yet uncertain.) Specifically, all ultraviolet divergences of the effective action cancel when we are along the fixed line. This is not to say however that the theory is divergence-free. Any field theory has ultraviolet divergent operator renormalizations; in our models these appear only for non-chiral operators. Of course, infrared divergences will be present also.

A. $SU(3)$ with $N_f = 9$

A simple candidate, first suggested in [3], is $SU(3)$ with nine fields Q^r in the fundamental representation and nine $\tilde{Q}_r$ in the antifundamental. Consider the superpotential

$$W = h \left(Q^1 Q^2 Q^3 + Q^4 Q^5 Q^6 + Q^7 Q^8 Q^9 + \tilde{Q}_1 \tilde{Q}_2 \tilde{Q}_3 + \tilde{Q}_4 \tilde{Q}_5 \tilde{Q}_6 + \tilde{Q}_7 \tilde{Q}_8 \tilde{Q}_9 \right) \ . \tag{13}$$

Though this superpotential breaks the $[SU(9)]^2 \ltimes \mathbb{Z}_2$ flavor symmetry down to $[SU(3)^3 \ltimes S_3]^2 \ltimes \mathbb{Z}_2$, the $Q^r, \tilde{Q}_s$ are still in an irreducible multiplet of the flavor symmetry and therefore all have the same anomalous dimension. The scaling coefficients $A_g \propto A_h = \frac{3}{2}\gamma$ both vanish along the curve $\gamma(g, h) = 0$, where the theory is ultraviolet finite.

Strictly speaking, the exactly marginal operator we have found is a linear combination of (13) and $W^2 \equiv \mathrm{tr} W_\beta W^\beta$, the square of the gauge chiral superfield, whose coefficients are determined by the equation $\gamma(g, h) = 0$. This will be true in all the models we present. Rather than state this repeatedly, we simply write the matter component (when it exists) of the marginal operator, leaving its gauge component implicit.

The importance of maintaining flavor symmetries should not be overlooked. For example, an operator of the form $(Q^1 Q^2 Q^3 + \tilde{Q}_1 \tilde{Q}_2 \tilde{Q}_3)$ is not marginal past leading order. The matter fields lie in a reducible multiplet under the $[SU(3) \times SU(6)]^2 \ltimes \mathbb{Z}_2$ symmetry, so the anomalous dimensions of the fields will not all be the same, and the scaling coefficients will no longer be proportional. This can easily be seen at the one-loop level.

B. $SU(3) \times SU(3)$ with $3 \times [(\mathbf{3},\mathbf{3}) + (\overline{\mathbf{3}},\overline{\mathbf{3}})]$

We now consider a model which has gauge group $SU(3) \times SU(3)$ with three flavors $Q^r, \tilde{Q}_u$ in the representations $(\mathbf{3},\mathbf{3})$, $(\overline{\mathbf{3}},\overline{\mathbf{3}})$ respectively. There are two marginal operators

$$h_1 \sum_{r=1}^3 ([Q^r]^3 + [\tilde{Q}_r]^3)$$
$$h_2(Q^1 Q^2 Q^3 + \tilde{Q}_1 \tilde{Q}_2 \tilde{Q}_3) \ . \tag{14}$$

Since the matter content of the two $SU(3)$ subgroups is the same, the scaling coefficients for the gauge couplings are the same; thus a third marginal operator is a linear combination of the field strength operators W_1^2 and W_2^2. The solutions of $\gamma(g_1, g_2, h_1, h_2) = 0$ form a three-dimensional manifold of fixed points which passes through zero coupling. This model is similar in appearance to a two-dimensional Landau-Ginsburg model. Normally a Landau-Ginsburg model of chiral superfields in four-dimensions is trivial; it is a Wess-Zumino type theory of scalars and fermions and the couplings in its superpotential are known to have positive β-functions. However, the addition of gauge interactions to the model stabilizes it against triviality by reducing the anomalous dimensions of the matter fields. Other models of this type will appear below.

C. $SU(2) \times SU(2)$ with a $(\mathbf{3},\mathbf{3})$

This model has an interesting connection with N=4 supersymmetry. Consider the marginal superpotential $W = hQ^3$. The one-loop gauge β-functions are zero; indeed, if we shut off one of the two gauge couplings and set h equal to the other, we have an N=4 model. As in the $SU(3) \times SU(3)$ model above, the vanishing of the three scaling coefficients occurs along a two-dimensional manifold in (g_1, g_2, h)-space; this manifold passes through the two N=4 fixed lines at $g_1 = h, g_2 = 0$ and $g_2 = h, g_1 = 0$. There is a strong-weak coupling duality symmetry along the N=4 lines; it would be interesting to understand whether it can be continued in some form (most likely acting only on the chiral operators of the theory) onto the entire manifold.

D. N=4 with a d-type coupling

Another finite model [3,4,6] is an $SU(N > 2)$ theory with the same matter content as an N=4 supersymmetric model but with a superpotential that combines the three adjoint superfields symmetrically rather than antisymmetrically. The superpotential

$$W = (h_1 \ f^{abc} + h_2 \ d^{abc})\Sigma_1^a \Sigma_2^b \Sigma_3^c \ , \tag{15}$$

where a, b, c are adjoint group indices and f, d are the antisymmetric and symmetric invariants of the group, has two independent couplings h_1 and h_2 which have proportional scaling coefficients. We may also add an independent operator

$$h_3 \ d^{abc} \sum_{i=1}^3 \Sigma_i^a \Sigma_i^b \Sigma_i^c \tag{16}$$

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Consequently, this N=1 model has a three-dimensional manifold, of which the N=4 line is a subset, specified by $\gamma(g, h_1, h_2, h_3) = 0$. Since there is expected to be a duality relating strong to weak coupling on the N=4 fixed line, it is tempting to conjecture, as in the previous case, that this duality extends in some non-trivial way over the entire manifold of N=1 fixed points.

E. An N=2 model with an additional operator

The previous example could be considered an N=2 model with two additional operators. Certain other finite N=2 models also contain additional exactly marginal operators. Consider an $SU(N)$ theory with fields $S^{\{\alpha\beta\}}, \tilde{S}_{\{\alpha\beta\}}$ in the symmetric tensor representations and fields $A^{[\alpha\beta]}, \tilde{A}_{[\alpha\beta]}$ in the antisymmetric tensor representations, along with the adjoint field Σ which is part of the N=2 gauge multiplet. If we consider a superpotential

$$W = h_A \tilde{A}\Sigma A + h_S \tilde{S}\Sigma S + y(\tilde{S}\Sigma A + \tilde{A}\Sigma S) \tag{17}$$

it is easy to see that there is a two-dimensional manifold of fixed points which contains the N=2 fixed line at $y = 0, h_A = h_S = g$. Assuming that this N=2 fixed line has an strong-weak coupling duality transformation, it would again be interesting to understand how it extends to the remainder of this manifold.

F. A chiral example

A simple chiral model [6] is E_6 with twelve fields in the **27** representation. Since three **27**'s may be combined to form an invariant, there are marginal operators of the form

$$h_1 \sum_{r=1}^{12} (Q^r)^3 \tag{18}$$

$$h_2 \left(Q^1 Q^2 Q^3 + Q^4 Q^5 Q^6 + Q^7 Q^8 Q^9 + Q^{10} Q^{11} Q^{12} \right) \ .$$

A slight modification of this model demonstrates an important point. If we replace $0 < k < 12$ of the **27** fields with $\overline{\mathbf{27}}$ fields, we no longer have a symmetry (except at $k = 6$) which ensures that all twelve fields have the same anomalous dimensions for arbitrary values of the couplings. Still, the operator

$$h \sum_{r=1}^{12-k} (Q^r)^3 + h' \sum_{s=1}^{k} (\tilde{Q}_s)^3 \tag{19}$$

is marginal. The scaling coefficients

$$\begin{aligned} A_g &= -(12 - k)\gamma_Q - k\gamma_{\tilde{Q}} \\ A_h &= \tfrac{3}{2}\gamma_Q \\ A_h' &= \tfrac{3}{2}\gamma_{\tilde{Q}} \end{aligned} \tag{20}$$

are linearly dependent, so we expect a fixed curve with non-vanishing and unequal h and h' on which all anomalous dimensions vanish. (Note that h and h' will be equal at leading orders in perturbation theory.) By contrast, the operator with $h \neq 0, h' = 0$ is not marginal, since A_g and A_h are linearly independent.

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G. An $SU(4)$ model with undetermined R charge

A more complicated example is found in an $SU(4)$ gauge theory with four antisymmetric tensors $A_a^{[\alpha\beta]}$, $\tilde{A}_{a[\alpha\beta]}$ and eight flavors of fundamentals, $Q^{r\alpha}$, $\tilde{Q}_{r\alpha}$. As an illustration of a subtlety, first consider the operator

$$W = h \sum_{r=1}^{4} \left(Q^r A_1 Q^{r+4} + Q^r A_2 Q^{r+4} + \tilde{Q}_r \tilde{A}_1 \tilde{Q}_{r+4} + \tilde{Q}_r \tilde{A}_2 \tilde{Q}_{r+4} \right). \tag{21}$$

This is only marginal at leading order. The fields A_1 and A_2 will mix through Q loops, and when the matrix of anomalous dimensions is diagonalized the combinations $A_1 + A_2$ and $A_1 - A_2$ will have different eigenvalues; in fact the latter does not couple to the Q fields at all.

Instead consider the marginal operator

$$W = h \sum_{i=1}^{2} \left(Q^{2i-1} A_1 Q^{2i+3} + Q^{2i} A_2 Q^{2i+4} + \tilde{Q}_{2i-1} \tilde{A}_1 \tilde{Q}_{2i+3} + \tilde{Q}_{2i} \tilde{A}_2 \tilde{Q}_{2i+4} \right). \tag{22}$$

This operator preserves sufficient symmetry to ensure that the anomalous dimensions of A and $\tilde{A}$ are all equal, as are those of Q and $\tilde{Q}$. The scaling coefficients are proportional

$$\begin{aligned} A_g &= -(4\gamma_A + 8\gamma_Q) \\ A_h &= \tfrac{1}{2}(\gamma_A + 2\gamma_Q) \end{aligned} \tag{23}$$

so the theory has a curve of fixed points. Here we have not shown that the anomalous dimensions all vanish; only the sum $\gamma_A + 2\gamma_Q$ must be zero on the fixed curve. It can be shown that all models of this type have anomalous dimensions which are zero at one-loop, but finiteness to all orders, in our view, has not been clearly established [4].

This situation, where the scaling coefficients vanish but the anomalous dimensions are not required to vanish, often arises. The ambiguity in the anomalous dimensions is equivalent to the statement that more than one gauge-anomaly-free R charge is present, and we do not know which R current appears in the same multiplet as the energy-momentum tensor. In this case, in addition to the R_0 charge present in the free theory, there is a charge X with $X(Q) = 1$, $X(A) = -2$ which is anomaly-free, so the dimensions of operators are given by the charge $R = R_0 + \frac{1}{3} X \gamma_Q$. The theory is finite if and only if $R = R_0$.

H. Other models with undetermined R charge

There are many other models for which our methods fail to prove finiteness to all orders, but which do have a fixed curve passing through zero coupling. For example, there is a series of models with $SO(N)$ gauge group ($N = 3 - 10, 12, 14, 18$) with both spinor and vector representations. These include a chiral $SO(10)$ theory [3] with eight "generations" of $\mathbf{16}$'s and $\mathbf{10}$'s, for which a marginal operator is $W = h \sum_{i=1}^{8} \mathbf{16}_i \mathbf{10}_i \mathbf{16}_i$.

A more intricate model [7] which was proposed as a GUT candidate has three generations of matter and Higgs superfields $\Psi_f, \Lambda_f, H_f^u, H_f^d$ ($f = 1, 2, 3$) in the $\mathbf{\bar{5}}, \mathbf{10}, \mathbf{5}, \mathbf{\bar{5}}$ representations, along with a field Σ in the $\mathbf{24}$ to break $SU(5)$ to the Standard Model gauge group, and two

extra chiral superfields $S, \tilde{S}$ in the $\mathbf{5}, \overline{\mathbf{5}}$ representations. Ignoring mass terms, the couplings in the superpotential are

$$\sum_{f=1}^{3} \left(h_1 \left[\Psi_f H_f^u \Lambda_f \right] + h_2 \left[\Lambda_f H_f^d \Lambda_f \right] \right) + h_3 S \tilde{S} \Sigma + h_4 \Sigma \Sigma \Sigma . \tag{24}$$

The gauge scaling coefficient

$$A_g = -3(\gamma_\Psi + 3\gamma_\Lambda + \gamma_{H^u} + \gamma_{H^d}) - \gamma_S - \gamma_{\tilde{S}} - 10\gamma_\Sigma \tag{25}$$

is proportional to $A_1 + A_2 + \frac{1}{3}A_3 + 3A_4$, where A_p is the scaling coefficient for h_p.

V. FROM WEAKLY TO STRONGLY COUPLED FIXED LINES

So far, we have discussed models which were in some sense generalizations of N=4 and N=2 models in that they contained trilinear couplings in the superpotential, leading to fixed curves that passed through zero coupling. In the remainder of the paper we will consider models for which, if a manifold of fixed points exists, it does not pass through the origin. In this section, we consider a class of models which interpolates between weakly and strongly coupled manifolds of fixed points. The existence of this set of fixed curves suggests that our approach can be applied to strong coupling fixed points. We begin with a model which has a weakly coupled large-N limit.

A. $Sp(2N) \times Sp(2N)$ with three $(\mathbf{2N}, \mathbf{2N})$

Consider a theory with gauge group $Sp(2N) \times Sp(2N)$ with coupling constants g_1, g_2. The matter fields consist of three multiplets transforming in the $(\mathbf{2N}, \mathbf{2N})$ representation. We note that if we take one of the gauge couplings to zero, we obtain an $Sp(2N)$ model with $6N$ multiplets transforming as $\mathbf{2N}$; this model, according to Ref. [8], has an interacting fixed point for all $N > 1$. In fact, in the large N limit, the fixed point is weakly coupled. Thus for large N we know that there are weakly coupled fixed points at $g_1 = g_*$, $g_2 = 0$ and at $g_2 = g_*$, $g_1 = 0$, where the matter fields have anomalous dimension $\gamma_* = -N^{-1}$. Furthermore, we expect there to be a curve of fixed points joining these points, as in Fig. 2, because, as both $Sp(2N)$ subgroups have identical matter content, the two scaling coefficients are equal.

$$A_{g_1} = A_{g_2} = -3(N+1) + 3N - 3N\gamma(g_1, g_2). \tag{26}$$

Thus, the vanishing of the scaling coefficients puts only one constraint on the two coupling constants. (One can see signs of this fixed curve at one loop; the anomalous dimension of the matter fields is $\gamma \approx -AN(g_1^2 + g_2^2)$, $A > 0$.) These results are reliable for large N, but since the proportionality of the scaling coefficients in (26) is true for any N, and since it is believed [8] that there are fixed points for $N > 1$ where one coupling vanishes, we argue that fixed curves exist for all N. (It is possible that, for small N, the fixed curves emanating from $g_1 = g_*$, $g_2 = 0$ and $g_2 = g_*$, $g_1 = 0$ are not connected.) The operator which is exactly marginal along the fixed curve generates a change in the coupling constants, so we expect it is a linear combination of W_1^2 and W_2^2. We cannot determine the two coefficients, except at $g_1 = g_2$ where the operator must be $W_1^2 - W_2^2$.

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B. $G \times G$ with $(\mathbf{R}, \mathbf{R})$ representations

The existence of the above class of models constitutes evidence that Eqs. (9) and (10) can be applied at strong coupling. We now generalize the previous case to include sets of theories which do not have a weak coupling extrapolation. For example, take a theory with any gauge group G and matter representations $\mathbf{R_i}$ which has a non-trivial fixed point. We conjecture that a theory with $G \times G$ and matter in representations $(\mathbf{R_i}, \mathbf{R_i})$ will then have a fixed curve; the scaling coefficients are still proportional since both G subgroups have the same matter content. Again, we expect that the marginal operator is something of the form $W_1^2 - C(g_1, g_2)W_2^2$, with $C(g_1, g_2)$ determined by the dependence of the anomalous dimensions on the coupling constants.

C. $SU(2) \times SU(2) \approx SO(4)$ with N_f flavors of $(\mathbf{2}, \mathbf{2}) \approx \mathbf{4}$

The case of $SU(2) \times SU(2)$ with $N_f > 3$ copies of $(\mathbf{2}, \mathbf{2})$ is particularly interesting. According to Ref. [8] this model has interacting fixed points for $N_f = 4, 5$. The arguments of [9] and of this section imply that these have fixed curves generated by $W_1^2 - C(g_1, g_2)W_2^2$.

Furthermore [9], the N=1 duality studied by Seiberg has a number of interesting implications. N=1 duality [8] maps this theory to $SO(N_f)$ with N_f vector multiplets; at $g_1 = g_2$ the operator $W_1^2 - W_2^2$ is mapped to the baryon operator of the dual theory $B_D = Q_D^{N_f}$. (For $N_f > 5$ the dual model is believed not to have a fixed point.) The fixed curve in the original theory is therefore apparently mapped to a fixed curve generated by the baryon. This suggests that the baryon in the dual theory is exactly marginal. One may easily confirm that this claim is consistent with the anomaly coefficients. On the other hand, for $N_f = 4$, where both the original and dual theory have gauge group $SO(4)$, the baryon $B = Q^4$ in the *original* theory is mapped to $W_{1D}^2 - W_{2D}^2$ in the *dual* theory, which generates a fixed curve. In summary [9] the case $N_f = 4$ has a two-dimensional manifold of fixed points generated at $g_1 = g_2$ by the exactly marginal operators $W_1^2 - W_2^2$ and Q^4. One may easily check that the scaling coefficients for the two gauge couplings and for the baryon operator are proportional.

$$A_{g_1} = A_{g_2} = (-12 + 8) - 8\gamma_Q$$
$$A_h = 1 + 2\gamma_Q \qquad (27)$$

This puts one condition on the three couplings, showing that this claim is consistent.

VI. NON-RENORMALIZABLE OPERATORS

In the case of $SO(4)$ discussed above an exactly marginal renormalizable operator was mapped under N=1 duality to an exactly marginal but perturbatively non-renormalizable operator. Initially one would have been reluctant to believe that a formula like (10) would apply to a non-renormalizable operator. However, the arguments of Sec. II A do not require that the theory have an ultraviolet fixed point, and apply to effective field theories. Still, they can break down as discussed in Sec. II B, and one might have worried that this would always happen. The existence of the N=1 duality map and the examples in Sec. V C suggests that, at least in some cases, the formulas (9) and (10) are appropriate to describe the physics.

Drawing confidence from the known examples, we now present other theories which have candidate marginal operators that are perturbatively non-renormalizable. In many models, more than one such operator can be found; we content ourselves with displaying a single example.

A. Some important issues

First, we must stress that in all of the cases presented below, the existence of strongly coupled manifolds of fixed points is merely conjecture, both because the dynamics of specific non-renormalizable theories might be more complex than or quite different from what we suppose, invalidating the arguments of Sec. II A, and because some of the models we present may not have any fixed points at all. Some of these theories were studied in [8] and arguments for fixed points were given; in these cases we have more confidence that the fixed curves that we conjecture are really present. Additional support for our approach stems from certain two-dimensional models where similar phenomena are known to occur [21].

The fact that these operators are not renormalizable should not by itself be cause for discarding our approach, as such theories do make sense as long as they are considered effective theories valid below some cutoff M_0. As we will see later, many of these non-renormalizable field theories are the low-energy effective expression of a renormalizable field theory, which may serve as a supersymmetric ultraviolet cutoff. This cutoff should not be taken to infinity. The non-renormalizable terms in the superpotential are irrelevant when the gauge coupling is weak, so any fixed point at which they are marginal must have a substantial gauge coupling. Since the renormalization group flow increases the gauge coupling logarithmically while suppressing the non-renormalizable couplings by powers, the scale at which the theory reaches its infrared fixed point should not lie too far below M_0 and the gauge coupling at M_0 should be finite.

In contrast to the renormalizable theories considered earlier, these models are subject to non-perturbative renormalizations. Suppose the superpotential has coupling constants h_s of negative dimension. In general there is a combination H of the h_s which, when multiplied by the dynamical scale of the theory Λ^{b_0}, is dimensionless and invariant under all global symmetries. The couplings h_s may then be multiplicatively renormalized by a holomorphic function $f(H\Lambda^{b_0})$. The function f may have singularities of various types, so the renormalized couplings $h_s^R = h_s f(H\Lambda^{b_0})$ may not be finite for all finite values of the bare couplings h_s. However, if any symmetries are restored at $h_s = 0$ (as is always the case in our examples) then h_s^R must be zero there also, from which it follows that the function h_s^R must be finite as $h_s = 0$ is approached along any direction. We therefore conclude that although h_s^R may be infinite for particular values of h_s, there is a neighborhood of the point $h_s = 0$ where the renormalized couplings are finite and the existence of a marginal operator with marginal coupling h_s^R may be established using our methods.

Almost all of the non-renormalizable superpotentials we study involve operators of dimension four or five. An occasional dimension six operator may arise. Beyond dimension six it is difficult to find a marginal operator. The reason has to do with unitarity and the dimensions of gauge invariant operators [19]; at a superconformal fixed point the dimensions of all non-trivial gauge invariant operators must be greater than or equal to one. If an operator of canonical dimension d is to be marginal, then, at the fixed point, the dimension

of at least one of the fundamental fields which it contains must be $d_\phi \leq 3/d$, which is less than $\frac{1}{2}$ if $d > 6$. It is difficult (though probably not impossible) to construct a situation where $d_\phi < \frac{1}{2}$ for some field ϕ, yet there are no gauge invariant bilinears of dimension less than one. In particular this can only happen in a chiral theory, since in a vector-like theory there will be a field $\tilde{\phi}$ (which may be ϕ itself) with $d_{\tilde{\phi}} = d_\phi$, giving a meson $\phi\tilde{\phi}$ of dimension less than one.

Even in the case of a candidate marginal operator of dimension six, there may be difficulties in a vector-like theory, since the meson $\phi\tilde{\phi}$ has dimension one and must therefore be free. Our methods often cannot determine whether or not a given operator is marginal in this case. An example which has a dimension six marginal operator will be given in Sec. VI C. As an example where a dimension six operator satisfies our conditions but fails to create a marginal deformation, consider a theory with gauge group $SO(6)$ and six vector representations. This theory has a candidate marginal baryon operator, which is mapped under N=1 duality [8,9] to the operator $W_1^2 - W_2^2$ in an $SO(4) \approx SU(2) \times SU(2)$ theory with six vector representations. When this operator is turned off, the $SO(6)$ theory flows to strong coupling, while the $SO(4)$ model is free in the infrared [8]. The operator $W_1^2 - W_2^2$ thus has no physical consequences in the $SO(4)$ theory, leading us to suspect that the marginal baryon operator in $SO(6)$ is an irrelevant perturbation of the strongly coupled theory.

B. Some $SU(4)$ examples

Amongst the $SU(N_c)$ interacting fixed points studied in [8], we find one example ($N_c = 4, N_f = 8$) with a marginal baryon operator. As in the $SU(3)$ example considered earlier, we need an operator which is marginal at the known infrared fixed point and which preserves enough flavor symmetry that all fields have the same anomalous dimension γ_Q. A suitable choice is

$$h(Q^1 Q^2 Q^3 Q^4 + Q^5 Q^6 Q^7 Q^8 + \tilde{Q}_1 \tilde{Q}_2 \tilde{Q}_3 \tilde{Q}_4 + \tilde{Q}_5 \tilde{Q}_6 \tilde{Q}_7 \tilde{Q}_8). \tag{28}$$

Both the gauge and Yukawa scaling coefficients are proportional to $1 + 2\gamma_Q$. The qualitative features of the renormalization group flow are illustrated in Fig. 3.

Up to now we have ignored the differences between the original and dual models under N=1 duality [8]. In particular we have disregarded the fact that the dual model contains singlets and a superpotential coupling them to the other fields. It is useful to focus our attention briefly on this point, though it will be seen that the effect of the singlets is minimal. In the case at hand the dual theory [8] has the same color and flavor content as the original, though the flavor representation of the fields is conjugate to that of the original theory. In addition to the colored fields $q_r, \tilde{q}^s$ there are color-singlet fields $M_s^{\ r}$ which are coupled through the superpotential $W = \lambda M_s^{\ r} q_r \tilde{q}^s$. From the operator mapping described in [8] we expect the marginal operator to be the same as the above with Q replaced by q. The scaling coefficients for the three coupling constants g, h, λ are

$$\begin{aligned}
A_g &= -4 - 8\gamma_q(g, h, \lambda) \\
A_h &= 1 + 2\gamma_q(g, h, \lambda) \\
A_\lambda &= \tfrac{1}{2}\gamma_M(g, h, \lambda) + \gamma_q(g, h, \lambda) \ .
\end{aligned} \tag{29}$$

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Since the first two are proportional (in fact they are the same as in the original model) there are two constraints on three couplings, and we expect a fixed curve emanating from the conjectured fixed point at $(g, h, \lambda) = (g_*, 0, \lambda_*)$. The coefficient A_λ simply sets $\gamma_M = 1$ on the fixed curve and is otherwise inert. We also learn that $\lambda_* = 0$ would not be a stable fixed point. At $\lambda = 0$ the singlets decouple from the theory and, were this a fixed point, their anomalous dimensions would have to vanish by unitarity. As a result A_λ would be negative (since $\gamma_q \approx -\frac{1}{2}$ there) so the theory would be driven away from $\lambda = 0$. We conclude that $\lambda_* \neq 0$.

Another very similar model has eight antisymmetric tensors; an exactly marginal operator is

$$h\epsilon_{\alpha\gamma\theta\kappa}\epsilon_{\beta\delta\eta\lambda}(A_1^{\alpha\beta}A_2^{\gamma\delta}A_3^{\theta\eta}A_4^{\kappa\lambda} + A_5^{\alpha\beta}A_6^{\gamma\delta}A_7^{\theta\eta}A_8^{\kappa\lambda}) \ . \tag{30}$$

Since this model is equivalent to $SO(6)$ with eight vector multiplets, which has a fixed point at $h = 0$ [8,9], we are confident that it has a fixed curve.

To illustrate the issues associated with models which lack a unique R charge, we present two more $SU(4)$ examples. The first has four $(\mathbf{4} + \overline{\mathbf{4}})$ representations $Q^r, \tilde{Q}^r$ and four $\mathbf{6}$ representations $A_{[\alpha\beta]}^r$; a suitable operator is

$$h\sum_{r=1}^{4} \tilde{Q}^r A^r A^r Q^r \ . \tag{31}$$

In this case the dimensions of operators are not determined. However, by adding another marginal operator

$$y_Q(Q^1 Q^2 Q^3 Q^4 + \tilde{Q}_1 \tilde{Q}_2 \tilde{Q}_3 \tilde{Q}_4) + y_A \mathrm{tr}(A^1 \tilde{A}_1 A^2 \tilde{A}_2 + A^3 \tilde{A}_3 A^4 \tilde{A}_4) \tag{32}$$

we can fix the dimensions of the fields. This is reminiscent of the situation in two-dimensional Landau-Ginsburg models.

Finally, consider a theory with fields $Q, \tilde{Q}, A, \tilde{A}, \Sigma$ in the $\mathbf{4}, \overline{\mathbf{4}}, \mathbf{6}, \mathbf{6}, \mathbf{15}$ representations. A candidate operator is

$$h\tilde{Q}\Sigma\Sigma Q + h'\tilde{A}\Sigma\Sigma\Sigma A \tag{33}$$

The scaling coefficients

$$\begin{aligned}
A_g &= -[5 + 4\gamma_\Sigma + \gamma_A + \gamma_{\tilde{A}} + \tfrac{1}{2}\gamma_Q + \tfrac{1}{2}\gamma_{\tilde{Q}}] \\
A_h &= 1 + \gamma_\Sigma + \tfrac{1}{2}\gamma_Q + \tfrac{1}{2}\gamma_{\tilde{Q}} \\
A_{h'} &= 2 + \tfrac{3}{2}\gamma_\Sigma + \tfrac{1}{2}\gamma_A + \tfrac{1}{2}\gamma_{\tilde{A}}
\end{aligned} \tag{34}$$

are linearly dependent. Unlike the previous case there is no marginal superpotential which can fix the dimensions of the fields.

C. $SU(6)$ with nine $(\mathbf{6} + \overline{\mathbf{6}})$ and singlets

The N=1 duality of Ref. [8] has interesting implications for this model, which is dual to that of Sec. IV A. The superpotential

$$\lambda M_s^r q_r \tilde{q}^s + h(B^{123} + B^{456} + B^{789} + \tilde{B}_{123} + \tilde{B}_{456} + \tilde{B}_{789}) \ , \tag{35}$$

where $B^{123} \propto q_4 q_5 q_6 q_7 q_8 q_9$, *etc.*, is a candidate marginal operator. According to Ref. [8], when $h = 0$ this model runs to infinite coupling and does not reach a fixed point. This corresponds to the fact that its dual, $SU(3)$ with nine flavors, runs to zero coupling. However, we showed earlier that in fact its dual has a fixed curve generated by the operator (13), which is dual [8] to the operator (35). It is therefore tempting to suggest that when $h \neq 0$ the $SU(6)$ model does not run all the way to infinite coupling, stopping instead on a fixed curve which runs off to infinite g as $h \to 0$, as in Fig. 4. The fact that the meson operator $q^r \tilde{q}_s$ has dimension one does not rule out this possibility, as it is redundant. If (35) is indeed a marginal operator, it is possible that semi-classical methods applied on the fixed curve of its weakly coupled dual can be used to study the N=1 duality transformation.

D. A Chiral Model

The strongly interacting models we have considered up to now are vector-like. Here we present a candidate chiral model: $SU(5)$ with five generations $A_i, \tilde{Q}_i$ of $\mathbf{10} + \overline{\mathbf{5}}$. Using $\overline{\mathbf{5}} \in \mathbf{10} \times \mathbf{10}$ (symmetric combination) and $\mathbf{5} \in \mathbf{10} \times \overline{\mathbf{5}}$, one can see there is an operator

$$h \sum_{i=1}^{5} A_i A_i A_i \tilde{Q}_i \tag{36}$$

which maintains the five-fold flavor symmetry. The index of the Q field is one while that of A is three; thus the scaling coefficients are proportional

$$\begin{array}{l} A_g = -5 - \frac{15}{2}\gamma_A - \frac{5}{2}\gamma_Q \\ A_h = 1 + \frac{3}{2}\gamma_A + \frac{1}{2}\gamma_Q \end{array} \tag{37}$$

and this model can have a fixed curve.

VII. RENORMALIZATION GROUP FLOW BETWEEN FIXED MANIFOLDS

In this section we study flow between fixed manifolds. In particular we would like to understand what happens when a field is given a mass and integrated out, or when the Higgs mechanism breaks part of the gauge symmetry. We will focus largely on the first case, showing first that manifolds of fixed points often flow to new ones when a theory is perturbed by a mass term, and then giving a number of examples. Finite theories with N=2 supersymmetry flow to a wide variety of interesting models. These include certain special cases studied in [8]; we identify certain suggestive properties of these models that seem to relate N=2 duality to N=1 duality. (We will return to this issue in Sec. VIII.) Finally we mention a couple of examples in which symmetry breaking causes flow from one fixed manifold to another.

A. Effect of integrating out a field

We will now prove the following lemma. Consider a theory at a fixed point with a marginal operator given by a polynomial superpotential

$$W = \sum_s h_s W^{(s)} \tag{38}$$

where each term $W^{(s)}$ is a gauge-invariant product of at least three fields. Suppose a mass term may be written for two of the fields ϕ, ϕ' (not necessarily different.) If each term $W^{(s)}$ contains either one factor of ϕ or one factor of ϕ', then, when the mass term $m\phi\phi'$ is added as a perturbation on the fixed point theory, either the theory does not find a new fixed point, or, if it does, that fixed point has a candidate marginal operator.

To prove this is straightforward. For simplicity we consider a superpotential with only two terms, of the form $W = h\phi X + h'\phi' X'$, where X and X' are multi-linear in superfields but contain neither ϕ nor ϕ'. (From this specific example, the proof is easily extended to the general case. We omit the details as they would generate more notation than insight.)

Consider the addition of the mass term $m\phi\phi'$ to the superpotential. When we integrate out these fields we should implement the equations

$$\begin{aligned}
\frac{\partial W}{\partial \phi} = hX = -m\phi' \\
\frac{\partial W}{\partial \phi'} = h'X' = -m\phi
\end{aligned} \tag{39}$$

as operator statements. The left-hand sides of these equations are by assumption independent of both ϕ and ϕ'. This leads to a new superpotential

$$W^{\text{new}} = -\frac{hh'}{m}XX' \tag{40}$$

We will now show that this theory has at least one candidate marginal operator.

Since the original theory had a marginal operator, we know that there is some linear combination of the scaling coefficients which is zero, which we may write as

$$cA_h + c'A_{h'} = A_g = -[b_0 + \sum_i T(R_i)\gamma_i] \tag{41}$$

The form of the superpotential and Eq. (10) imply that $A_h = \frac{1}{2}\gamma_\phi + \cdots$, $A'_h = \frac{1}{2}\gamma_{\phi'} + \cdots$. From the previous equation and the existence of the mass term $m\phi\phi'$, which implies $T(R_\phi) = T(R_{\phi'})$, it follows that $c = c' = -2T(R_\phi)$. The new interaction is of the form HXX' where $H = -hh'/m$. Its scaling coefficient is

$$A_H = 1 + (A_h - \frac{1}{2}\gamma_\phi) + (A_{h'} - \frac{1}{2}\gamma_{\phi'}) \tag{42}$$

Using Eqs. (41)–(42),

$$cA_H = A_g - T(R_\phi)(1 - \gamma_\phi) - T(R_{\phi'})(1 - \gamma_{\phi'}) . \tag{43}$$

Following the integrating out of ϕ and ϕ', the new gauge scaling coefficient is

$$A_g^{new} = A_g - T(R_\phi)(1 - \gamma_\phi) - T(R_{\phi'})(1 - \gamma_{\phi'}) = cA_H \tag{44}$$

which indicates that the new superpotential contains one or more candidate marginal operators.

The new scaling coefficients put some constraints on the anomalous dimensions which must be satisfied at a new fixed point. We could have gotten the same constraints in another way. The mass term $m\phi\phi'$ broke the R symmetry which was satisfied at the old fixed point, but it conserves a new R symmetry. The linearity of the superpotential in ϕ and ϕ' ensures both that a new R exists and that unitarity is not violated by the presence of zero or negative dimension fields. Requiring the new R charge to be conserved classically and quantum mechanically puts conditions on the anomalous dimensions which are the same as those from the anomaly coefficients. One may see this more clearly by keeping the fields ϕ and ϕ' in the theory, treating the mass term as a new interaction, and rederiving the lemma. Once at the new fixed point, integrating out the fields ϕ and ϕ' does not change the conserved R charge or the dimensions of chiral operators. In particular, an operator is marginal whether expressed in terms of the fields ϕ and ϕ' or in terms of the other fields using (39). This will be important later.

We note also that the lemma applies to a wider class of superpotentials. If, to the superpotential W above, we add terms W' that are independent of ϕ and ϕ', then the low energy superpotential will be $W^{\text{new}} + W'$. If $W + W'$ has a marginal operator, and the fields in W' can be assigned new R charges such that $W^{\text{new}} + W'$ has a conserved R symmetry, then the low-energy theory will have a candidate marginal operator.

Before applying these ideas we make some general observations. A candidate operator may fail to be marginal under various circumstances—if it is a composite of redundant operators, if it preserves an R symmetry which does not characterize any fixed point of the theory, or if its canonical dimension is too high, preventing the putative fixed point from being unitary. The last of these reasons implies that the process of integrating out fields will terminate fairly quickly, as unitarity comes into question when dimension six operators begin to appear.

When we derive a non-renormalizable superpotential from a finite theory, the latter serves as an ultraviolet regulator for the former, ensuring that non-renormalizability of the effective theory does not generate uncontrollable infinities while leaving intact the infrared properties which we are studying. The existence of a sensible regulator should in our view allow the arguments of Sec. II A to be applied near a previously established fixed point, subject to the limitations described in Sec. II B.

B. Application to $SU(3)$ with $N_f = 9$

We may immediately find some new non-renormalizable candidate operators by turning to some finite models and integrating out selected fields. For example, if we take the model of Sec. IV A and add a mass term $mQ^9\tilde{Q}_9$, we generate a theory with eight flavors and superpotential

$$W = h\left(Q^1Q^2Q^3 + Q^4Q^5Q^6 + \tilde{Q}_1\tilde{Q}_2\tilde{Q}_3 + \tilde{Q}_4\tilde{Q}_5\tilde{Q}_6\right) + H(Q^7Q^8\tilde{Q}_7\tilde{Q}_8) \, . \tag{45}$$

Writing γ_Q for the first six flavors and $\gamma_{\hat{Q}}$ for the last two, the scaling coefficients are

$$A_g = -[1 + 6\gamma_Q + 2\gamma_{\hat{Q}}]$$
$$A_h = \tfrac{3}{2}\gamma_Q \tag{46}$$
$$A_H = 1 + 2\gamma_{\hat{Q}}$$

which shows that this superpotential is a candidate marginal operator. However, the flavor symmetry of this model does not treat the eight quarks symmetrically — the anomalous dimensions $\gamma_Q = 0$ and $\gamma_{\hat{Q}} = -\frac{1}{2}$ are not equal — and thus this fixed curve cannot be continued to $h = H = 0$, where the full $SU(8) \times SU(8)$ flavor symmetry would be discontinuously restored. Therefore, if a fixed curve is generated by this operator, it does *not* pass through the fixed point found in [8] in the absence of a superpotential. Of course there simply may be no fixed curve of this type.

Now let us add $m(Q^3\tilde{Q}_3 + Q^6\tilde{Q}_6 + Q^9\tilde{Q}_9)$. The resulting superpotential

$$W = H(Q^1Q^2\tilde{Q}_1\tilde{Q}_2 + Q^4Q^5\tilde{Q}_4\tilde{Q}_5 + Q^7Q^8\tilde{Q}_7\tilde{Q}_8) \tag{47}$$

is obviously a candidate marginal operator. Furthermore, this operator does preserve the symmetry among all six flavors and can generate a marginal deformation of the fixed point with $N_c = 3, N_f = 6$ studied in [8]. Thus, unlike the previous case, there is some evidence for the existence of a fixed curve in this model.

C. Application to a chiral model

We may find similar operators in some chiral theories. Taking as a starting point the finite $SO(10)$ model with eight "generations" of $\mathbf{16} + \mathbf{10}$ and marginal operator $h\sum_{i=1}^{8} \mathbf{16}_i\mathbf{10}_i\mathbf{16}_i$, we may integrate out all the $\mathbf{10}$ fields. Different operators emerge depending on how this is done. A diagonal mass term $m\sum_{i=1}^{8} \mathbf{10}_i\mathbf{10}_i$ generates a candidate marginal operator

$$H \sum_{i=1}^{8} \mathbf{16}_i\mathbf{16}_i\mathbf{16}_i\mathbf{16}_i \tag{48}$$

D. Application to an $SU(4)$ model

In Sec. IV G we studied a model of four antisymmetric tensors and eight flavors in the fundamental representation; this was a finite model whose marginal operator was given in (22). Mass terms for the antisymmetric tensor fields lead to a theory with eight flavors, studied in [8] and in Sec. VI B, that has various operators, of the form $Q^4, (Q\tilde{Q})^2, \tilde{Q}^4$, which can be assembled into exactly marginal combinations such as Eq. (28). Mass terms $mQ^r\tilde{Q}_r$ for $r = 5, 6, 7, 8$ lead to another theory studied in Sec. VI B; the resulting superpotential

$$H \sum_{i=1}^{2} \left(Q^{2i-1}A_1\tilde{A}_1\tilde{Q}_{2i-1} + Q^{2i}A_2\tilde{A}_2\tilde{Q}_{2i} \right) \tag{49}$$

differs only slightly from the operator presented in Eq. (31).

E. Application to finite N=2 models

The applications to N=2 models are wide-ranging and interesting. Finite N=2 models with a single type of group representation were discussed in Sec. III. Here we reconsider these along with theories involving more complicated matter representations. Since many or all of these N=2 models may have some form of duality, the N=1 models they flow to may also display duality of some type. In particular we will see evidence below that the N=1 duality of [8] is related to N=2 duality. This will be further explored in Sec. VIII.

First consider the model of Sec. IV D, which has the matter content of an N=4 model and three marginal operators. When we add a mass term $m(\Sigma_3)^2$ and integrate out one of the superfields in the presence of the superpotential (15) we find a low-energy theory with two adjoint superfields Σ_1 and Σ_2 and a superpotential made up of the three candidate marginal operators

$$(H_1 f^{abc}f^{ade} + H_2 f^{abc}d^{ade} + H_3 d^{abc}d^{ade})\Sigma_1^b\Sigma_2^c\Sigma_1^d\Sigma_2^e \tag{50}$$

If we add the operator (16) to the high-energy superpotential, then the lemma of Sec. VII A does not apply. In particular there is no conserved R charge when the mass term is added, so no fixed point can be reached until certain couplings have flowed to zero.

Next consider a finite N=2 model of $SU(4)$ with three anti-symmetric tensor representations $(A^i, \tilde{A}_i)$ and two fundamental representations $(Q^r, \tilde{Q}_r)$. To find a candidate marginal operator we may either add a mass term $m\,\text{tr}\Sigma^2$ or some combination of masses for the matter fields. A variety of different theories result. For example, if we add

$$m\left(\tilde{A}_1 A_2 + \tilde{A}_2 A_3 + \tilde{Q}_1 Q^2\right) \tag{51}$$

we arrive at the last model in Sec. VI B and its candidate marginal operator (33).

An interesting set of theories are the finite N=2 models involving only hypermultiplets in the defining representation. For example, consider $SU(N)$ with $2N$ hypermultiplets $(Q^r, \tilde{Q}_r)$. We may add mass terms of a number of types. By giving flavor off-diagonal masses to some of the hypermultiplets, one arrives at superpotentials which are a sum of operators such as $\tilde{Q}(\Sigma)^k Q$; special cases are those where all but N/p of the hypermultiplets are integrated out, for which the superpotential consists only of terms of the form $\tilde{Q}(\Sigma)^p Q$. Of course unitarity is lost for $p > 4$.

The most intriguing option, however, is to give a mass m_Σ to the adjoint field Σ and integrate it out. We then arrive at a model which was studied in Ref. [8], $SU(N_c)$ with $2N_c$ flavors, along with a superpotential

$$W = -h(Q_\alpha^r \tilde{Q}_r^\beta)\,T_\beta^{a\alpha}T_\delta^{a\gamma}\,(Q_\gamma^s \tilde{Q}_s^\delta) = -\frac{h}{2}\left[(Q_\alpha^r \tilde{Q}_s^\alpha)(Q_\beta^s \tilde{Q}_r^\beta) - \frac{1}{N_c}(Q_\alpha^r \tilde{Q}_r^\alpha)(Q_\beta^s \tilde{Q}_s^\beta)\right] \tag{52}$$

where T^a are color group matrices. We have used an $SU(N_c)$ group identity to rewrite the superpotential in terms of gauge singlets. This operator is exactly marginal. Having written it in this form it is natural to rewrite it using an auxiliary meson field N_r^s of canonical dimension two.[2]

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[2]We thank K. Intriligator for a discussion which led to this line of thought.

$$W = \lambda N_r^s (Q_\alpha^r \tilde{Q}_s^\alpha) + \frac{\lambda^2}{2h} \left[N_r^s N_s^r - \frac{1}{N_c} N_r^r N_s^s \right] \tag{53}$$

Again, all we have done here is rewrite the operator (52) in a new way. (A similar mechanism has been used in [9].) As $h \to 0$, the coefficient of the marginal operator (52) vanishes and the theory reduces to the original $N_f = 2N_c$ model studied in [8]. As $h \to \infty$, the meson mass term in (53) vanishes and the theory becomes remarkably similar to the N=1 dual [8] of the $N_f = 2N_c$ model. It seems that the fixed curve generated by the operator (52), (53) connects one theory to the other.

Since the small h theory is found by taking the gauge coupling in the N=2 theory to be small, we may expect that the when the N=2 gauge coupling is large the low energy theory is in the large h region. It is natural to suggest that the weak-strong coupling duality (S-duality) of the N=2 theory translates into some form of duality in the N=1 theory.

This naive picture is far too simplistic, and we provide a more careful though still incomplete analysis of the situation in Sec. VIII. At this stage we simply note that the same approach can be used for $SO(N)$ and $Sp(2N)$ gauge groups; the insertion of the auxiliary meson field allows one to rewrite the marginal operator in a form which suggests that the curve of fixed points that it generates connects the low-energy theory to its N=1 dual, or at least to a theory very similar to its dual.

F. Flow under symmetry breaking

Because of the changes in the representation content of a theory when gauge symmetries are broken, including the appearance of gauge singlet fields which we have largely avoided in this paper, the renormalization group flow associated with symmetry breaking is a rather complicated subject. It deserves a more systematic study than we have given it here. Under many circumstances, symmetry breaking leads a theory with a marginal operator to flow to a theory with a marginal operator of lower dimension. In general this occurs, as in the case of integrating out a massive field, when the symmetry breaking preserves a unitary and anomaly-free R charge consistent with the new flavor symmetries.

A first example involves the $SO(N)$ models with N vector representations, studied in Sec. V C. The baryon in $SO(5)$ with five flavors generates a curve of fixed points. As noted in [9], if we break the gauge symmetry to $SO(4)$ and then to $SO(3)$ this fixed curve flows to the fixed curves generated by the baryons of $SO(4)$ and $SO(3)$ respectively. The latter theory is N=4 supersymmetric and its fixed line passes through zero coupling.

Another more complicated example involves $SU(3N)$ with $2N$ flavors and an adjoint field. The superpotential

$$h \sum_{r=1}^{2N} Q^r \Sigma\Sigma\Sigma \, \tilde{Q}_r \tag{54}$$

is a candidate marginal operator. As discussed in the previous section, the finite N=2 model with gauge group $SU(3N)$ and $6N$ flavors flows to this model when $4N$ of the flavors are integrated out. Now give vacuum expectation values to $Q^r \tilde{Q}_r$ for $r = 1, \ldots, 2N$. This breaks the gauge symmetry to $SU(N)$. The fields $Q^r, \tilde{Q}_r$ are eaten by the broken gauge

fields, except for some neutral Higgs bosons. The field Σ transforms under the unbroken $SU(N)$ symmetry as

$$\text{adjoint} \rightarrow \text{adjoint} + \mathbf{N} \times 2N + \overline{\mathbf{N}} \times 2N + \mathbf{1} \times 4N^2 \tag{55}$$

Thus there are a number of singlet fields, along with the matter content of a finite N=2 $SU(N)$ model. A number of couplings appear in the superpotential. The conditions for a marginal operator have a unique solution, where all anomalous dimensions vanish. From unitarity considerations, a gauge singlet with vanishing anomalous dimension must decouple, so all couplings involving singlets flow to zero. The remaining superpotential is that of an N=2 theory along with a $y\Sigma^3$ interaction. From the scaling coefficients there can be only a one-dimensional manifold of fixed points; we therefore expect y to flow to zero and the theory to be driven onto the N=2 fixed line. From a one-loop computation one may confirm that this is the only possibility at weak coupling. Thus, one can flow from an N=2 fixed line to the fixed curve of this model by integrating out fields, and from this model down to another N=2 fixed line by breaking gauge symmetries.

Finally, we note that the example of Sec. VI C has the property that when the gauge symmetry is broken to $SU(3)$, leaving six flavors, the theory is related under N=1 duality to a model with the same color and flavor groups but with no singlet fields. We have seen that the theory without singlets has a marginal operator (47). In the broken $SU(6)$ model, the dual of the operator (47)

$$(M_1^1 M_2^2 - M_2^1 M_1^2) + (M_4^4 M_5^5 - M_5^4 M_4^5) + (M_7^7 M_8^8 - M_8^7 M_7^8) \tag{56}$$

is generated by instanton effects [22]. One may easily check that when combined with the classical superpotential $W = M_s^r q_r \tilde{q}^s$ it represents a marginal operator in this description.

VIII. MESON MASS OPERATORS AND N=1 DUALITY

In Sec. VII E we showed that there was a tantalizing connection between certain finite N=2 models and those N=1 theories studied by Seiberg [8] which have the same gauge group in both the original and dual descriptions. (A suggestion that such a connection might exist was made in [8] and related connections are present in [9].) We considered a finite N=2 model with gauge group $SU(N_c)$ and $2N_c$ hypermultiplets in the fundamental representation. We added a mass m_Σ for the adjoint chiral field, leading to a low-energy theory that we will call the N=$\cancel{2}$ model. We observed that the fixed line of the N=2 model flows to the infrared fixed curve of the N=$\cancel{2}$ model. Examining the superpotential (52) and its alternate form (53), we noted the similarity of certain limits of the N=$\cancel{2}$ model to the original and dual $SU(N_c)$ theories with $N_f = 2N_c$ studied in [8].

In this section we present speculative but, we hope, plausible arguments that the N=1 duality of [8] is closely associated with S-duality of finite N=2 models. We will find a theory, which we will call supersymmetric quantum chromesodynamics (SQCMD), that has the same infrared fixed curve as the N=$\cancel{2}$ model. We will learn more about the fixed curve of N=$\cancel{2}$ by studying SQCMD in detail. Eventually we will use conjectures about N=2 duality to guess its relationship to N=1 duality. We cannot verify our guess because we do not at this time have enough information about duality in the relevant N=2 theories.

A. Supersymmetric quantum chromesodynamics (SQCMD)

Consider a theory of $SU(N_c)$ with $2N_c$ flavors $Q^r, \tilde{Q}_s$ in the $\mathbf{N_c}$ and $\overline{\mathbf{N}}_\mathbf{c}$ of color, along with massive propagating (*not* auxiliary) singlet mesons N_s^r. We will give the mesons a slightly unusual mass term, and couple them to the other fields; the superpotential is

$$W = \lambda N_r^s Q^r \tilde{Q}_s + \frac{1}{2} m_0 \left[N_r^s N_s^r - \frac{1}{N_c} N_r^r N_s^s \right] \tag{57}$$

We have seen this superpotential in (53) during our study of the N=$\not2$ model, which contains auxiliary fields N_s^r. The SQCMD model has an anomaly-free $U(2N_c) \times U(1)_R$ flavor symmetry. Note that in the limits $m_0 = 0$ and $m_0 = \infty$ the flavor symmetry is enhanced to $SU(2N_c) \times SU(2N_c) \times U(1) \times U(1)_R$.

There is a unique flavor-independent gauge-anomaly-free R charge with $R(Q) = R(\tilde{Q}) = \frac{1}{2}$ and $R(N) = 1$. This charge determines the dimensions of chiral operators at any interacting fixed point. The theory with $m_0 = \infty$ is N=1 SQCD with $N_f = 2N_c$; according to [8] it flows to an interacting superconformal fixed point. The operator

$$\left[(Q_\alpha^r \tilde{Q}_s^\alpha)(Q_\beta^s \tilde{Q}_r^\beta) - \frac{1}{N_c}(Q_\alpha^r \tilde{Q}_r^\alpha)(Q_\beta^s \tilde{Q}_s^\beta) \right] \ , \tag{58}$$

seen earlier in (52) during our study of the N=$\not2$ model, is exactly marginal at this fixed point; its scaling coefficient is $A_h = 1 + 2\gamma_Q$ while the gauge scaling coefficient is $A_g = -(N_c + 2N_c\gamma_Q)$.

Of course, the N=$\not2$ model, in the limit in which the N=2 gauge coupling is taken to zero and m_Σ is taken to ∞, also becomes the N=1 SQCD theory studied in [8]; it flows to the same fixed point, and has the same marginal operator in the infrared, as the SQCMD model with $m_0 = \infty$.

The operator (58) generates a complex curve of fixed points. We may flow to this curve in two ways. One way is to take the N=$\not2$ model with finite N=2 gauge coupling τ, as seen in (52). The other is to let the mass m_0 in SQCMD be finite but large; integrating out the meson we generate the operator (58). The fact that in the ultraviolet the meson is an auxiliary field of dimension 2 in the N=$\not2$ model and a propagating canonical field of dimension 1 in SQCMD is unimportant; by the time the two theories have reached the infrared, the dimension of each meson has flowed to $\frac{3}{2}$ and the two theories are indistinguishable.

Generally, we do not know what value of h will be found at the low-energy fixed point when flowing from the N=$\not2$ or SQCMD theories with given initial values of τ or m_0. There is one exception: when $\tau \to i\infty$ or $m_0 = \infty$, then $h = 0$. However, we also know, by holomorphy, that continuous variations in τ or m_0 will generically lead to continuous variations in h, so for sufficiently large values of Im τ or of m_0, the low-energy coupling h will be small.

The structure of the fixed curve may be explored using SQCMD. Our key point is the following. From $m_0 = \infty$ the mass of the meson may be continued to zero without the theory leaving the fixed curve. This follows from the fact that the meson has $R = 1$, so that its mass preserves the R symmetry; thus the theory has the same R symmetry in the infrared independent of whether m_0 is zero, finite, or infinite. This is crucial. In general, a mass term is a relevant perturbation on a fixed point and causes the theory to flow to a new one. Here it is marginal, and the infrared theory remains superconformal for any m_0. There

are two distinct limits. When m_0 is much larger than the dynamical scale Λ of the theory, it is appropriate to integrate out the meson and think of the theory as SQCD with the superpotential (58). When $m_0 \ll \Lambda$, the meson mass is negligible in the ultraviolet, while in the infrared it becomes a dimensionless coupling constant — it undergoes dimensional anti-transmutation. In this case the meson should remain in the theory and the superpotential (57) is appropriate. But there is no dividing line between these descriptions and no reason whatever that they should not go smoothly into one another. We therefore conclude that the mass m_0 parameterizes a complex curve of fixed points which connects the theory with $m_0 = 0$ to that with $m_0 = \infty$. Note that we have not demanded that this complex curve (of real dimension two) be everywhere non-singular. We will require only that there is a smooth path (of real dimension one) connecting $m_0 = \infty$ ($h = 0$) to $m_0 = 0$.

We have noted that the theories at infinite and zero m_0 have enhanced flavor symmetries. Could they be the same point? The answer is no. The gauge invariant operators of the two theories do not match, as can be seen from consideration of [8]. The baryons

$$B^{r_1 \cdots r_N} \equiv \epsilon_{\alpha_1 \cdots \alpha_N} Q^{r_1 \alpha_1} \cdots Q^{r_N \alpha_N} \ , \tag{59}$$

with unit baryon number, exist all along the fixed curve and are the same operators at both endpoints, as are the antibaryons. At $m_0 = \infty$, the gauge invariant mesons are $Q^{r\alpha}\tilde{Q}_{s\alpha}$ which have zero baryon number and transform as $(\mathbf{2N_c}, \overline{\mathbf{2N_c}})$ under the $SU(2N_c) \times SU(2N_c) \times U(1)$ flavor symmetry. For finite m_0 the flavor symmetry is broken by the superpotential to $U(2N_c)$; the operators $Q^r\tilde{Q}_s$ and N_r^s, both of which transform as a $(\mathbf{2N_c} \otimes \overline{\mathbf{2N_c}})$, are mixed by the equations of motion. At $m_0 = 0$, the larger flavor symmetry is again present; $Q^r\tilde{Q}_s$ is a redundant operator, and the only other gauge and baryon singlets are N_r^s, which transform as a $(\overline{\mathbf{2N_c}}, \mathbf{2N_c})$ under the flavor symmetry. No manipulation of the flavor symmetries can simultaneously make the baryons and mesons of the two theories match.

We have already seen that the limit $\tau \to i\infty$ (weak coupling) in the N=2 model leads to the same fixed point as does SQCMD with $m_0 = \infty$. What point does $m_0 = 0$ correspond to? It must be a special point in the N=2 model, since it has an enhanced flavor symmetry. As the adjoint field is flavor-blind, we expect that such a point must derive from a special point in the N=2 theory. Only the free theory $\tau \to i\infty$ is known to have an enhanced $SU(2N_c) \times SU(2N_c) \times U(1)$ flavor symmetry; at generic values of τ the N=2 model has merely a $U(2N_c)$ symmetry. A reasonable guess is that the theory has enhanced symmetry at infinite coupling ($\tau \to 0$), a point which should be related by S-duality to a free theory (of magnetic matter) which would have an enhanced flavor symmetry due to the absence of interactions.

We end this section by summarizing our conclusions, depicted in Fig. 5. We have found that the fixed curve of the N=2 model is the same as that generated by the meson mass operator (57) in SQCMD. There exists a path along which one may go smoothly from $m_0 = \infty$ to $m_0 = 0$; the endpoints of this path are inequivalent, though they share the same enhanced flavor symmetry. These points may also be reached from the N=2 model, the former in the limit of weak coupling, the latter in the limit of strong coupling.

B. N=2 duality and its effects

We now consider the action of S-duality on these theories. Much of what follows relies on several assumptions about its effects in N=2 theories for $SU(N_c > 2)$, since the duality of these theories has not yet been fully described. These assumptions are in part tailored to assure a relation between N=1 and N=2 duality, so we are not proving anything here. Still, our assumptions are consistent with the known case [20] and with other properties of these models, and we believe that our assumptions are at least in part correct.

The main assumption is that these finite models have duality under $\tau \leftrightarrow -1/\tau$. Let us consider what form this $\mathbb{Z}_2$ symmetry could take. It is clear the dual theory must also be a finite N=2 theory. However, of all these theories, only the $SU(N_c)$ model has a $U(2N_c)$ flavor symmetry for arbitrary N_c. [The flavor symmetries of finite $SO(N_c)$ and $Sp(2N_c)$ models with vector multiplets are $Sp(2N_c - 4)$ and $SO(4N_c + 4)$.] We therefore infer that the $SU(N_c)$ theory must be transformed into another theory with the same gauge and matter content. This is of course true in the known case of $SU(2)$ [20]. It is also encouraging that the semi-classical monopoles of the broken $SU(N_c)$ theory include states in the $\mathbf{2N_c}$ and $\overline{\mathbf{2N_c}}$ of flavor. There are also other monopoles in larger flavor representations; these states cannot be present in the unbroken N=2 theory as they would carry color and would give the theory a positive β-function.

In the $SU(2)$ case [20] the massless particles at the origin include the eight massless monopoles in the spinor of $SO(8)$. We will guess that the generalization of this statement is that there are monopole-like states q_r and $\tilde{q}^s$ in the $\mathbf{2N_c}$ and $\overline{\mathbf{2N_c}}$ of flavor which are massless in the unbroken theory. The need for a vanishing one-loop β-function forces us to put these fields into representations of index one; we will take q_r in the $\mathbf{N_c}$ of the dual $SU(N_c)$ and $\tilde{q}^s$ in the conjugate representation.

We must next consider the operator mapping in the N=2 model under S-duality. We begin at an arbitrary coupling τ, where the flavor symmetry is $U(2N_c)$. As in the N=2̸ model the gauge invariant operators include meson and baryon operators built from the invariant tensors of $SU(N_c)$. These same operators must reappear in the dual description of the theory.

Of course, the simplest way for this to occur would be for the fields $Q^r, \tilde{Q}_s$ to be mapped to the fields $\tilde{q}^r, q_s$, and for each operator built out of Q^r fields to be mapped to the same operator built out of $\tilde{q}^r$ fields. In this case the duality would simply map $\tau \to -1/\tau$ and leave the flavor representations unchanged. Also, $\tilde{q}^r$ would have positive baryon number. However, this hypothesis would lead to an inconsistency with a property of SQCMD discussed in the previous section. If there were no transformation of the flavor structure under N=2 duality, then the fixed line of the N=2 model would be dual to itself, with the theory at τ identical to the theory at $-1/\tau$. In this case the fixed curve of the N=2̸ model would be dual to itself; but we have already argued that the theories derived from $\tau = 0$ and $\tau = i\infty$ were inequivalent using the properties of SQCMD. We therefore discard the hypothesis that the flavor structure is unchanged under duality.

Since the mapping $\tau \to -1/\tau$ is a $\mathbb{Z}_2$ transformation, any other hypothesis must involve a $\mathbb{Z}_2$ transformation of the flavor representations. In this case the N=2 theory is mapped under $\tau \to -1/\tau$ to an inequivalent but similar N=2* theory, with a fixed line that contains the same physics except conjugated by a $\mathbb{Z}_2$. A natural candidate involves a $\mathbb{Z}_2$ automorphism

of $U(2N_c)$ — charge conjugation — which maps $SU(N_c)$ invariants as follows.

$$Q^{r_1} \cdots Q^{r_{N_c}} \leftrightarrow \epsilon^{r_1 \cdots r_{N_c} s_1 \cdots s_{N_c}} q_{s_1} \cdots q_{s_{N_c}}$$

$$\tilde{Q}_{s_1} \cdots \tilde{Q}_{s_{N_c}} \leftrightarrow \epsilon_{s_1 \cdots s_{N_c} r_1 \cdots r_{N_c}} \tilde{q}^{r_1} \cdots \tilde{q}^{r_{N_c}}$$

$$Q^r \tilde{Q}_s - \frac{1}{2N_c} \delta^r_s Q^u \tilde{Q}_u \leftrightarrow - \left[q_s \tilde{q}^r - \frac{1}{2N_c} \delta^r_s q_u \tilde{q}^u \right] \tag{60}$$

$$Q^r \tilde{Q}_r \leftrightarrow + q_r \tilde{q}^r$$

Under this transformation the q_r have positive baryon number. Note that the operator map cannot include the adjoint superfield Σ by symmetries and dimensional analysis.

Furthermore, the operator Σ^2 must map (again by symmetries and dimensional analysis) to itself under duality, so a Σ mass term in the N=2 theory will map to one in the N=2* theory. It follows that the N=2̸ model will be mapped to an inequivalent N=2̸* model under S-duality. Of course, the N=2 supersymmetry is broken when the adjoint field is massive, so we expect N=2 duality to be broken also. However, it is reasonable to expect, since N=1 supersymmetry is still preserved, that these violations of duality will occur in the Kähler potential, and that the duality of the superpotential and of the chiral operators will survive into the N=2̸ and N=2̸* models. This means that their infrared fixed points should be dual, and that the operator map (60) should be appropriate for the N=2̸ and N=2̸* theories.

Although we do not know the exact map between the couplings h and h^* (since both coupling constants can be non-perturbatively renormalized), we do know, from the enhanced flavor symmetries, that the point $h = 0$ on the N=2̸ fixed curve must be mapped to a special point $h^* \neq 0$ on the N=2̸* fixed curve. A natural guess is that the special point is at $h^* = \infty$. Similarly the point $h^* = 0$ would be mapped under S-duality to $h = \infty$. Thus, qualitatively, the small and large coupling regions exchange places. In terms of the auxiliary meson introduced in (53), the large mass region of the N=2̸ model is dual to the small mass region of the N=2̸* model, with the points $m_0 = 0, \infty$ exchanged with $m_0^* = \infty, 0$. This qualitative picture is shown in Fig. 6.

C. SQCMD and N=1 duality

By now the reader can clearly see where we are heading. The map between the N=2̸ and N=2̸* fixed curves induces a map between the SQCMD theory studied earlier and a dual theory SQCMD*. The latter has fields $q_r, \tilde{q}^s$ in the fundamental and antifundamental representations of $SU(N_c)$ and singlet mesons M^r_s of mass m_0^*. The map between the theories is not applicable in the ultraviolet; it is merely a map between their infrared fixed points.

Now consider a point on the fixed curve of SQCMD where m_0 is large. This point is mapped to a point with small m_0^* on the SQCMD* fixed curve. For large m_0 it is appropriate to integrate out the meson field, leaving the superpotential (58) with a small coupling h. The gauge invariant chiral operators of the theory are

$$Q^{r_1} \cdots Q^{r_{N_c}}, \quad \tilde{Q}_{s_1} \cdots \tilde{Q}_{s_{N_c}}, \quad Q^r \tilde{Q}_s . \tag{61}$$

The dual theory has superpotential

$$W_* = \lambda^* M_s^r q_r \tilde{q}^s + \frac{1}{2} m_0^* \left[M_r^s M_s^r - \frac{1}{N_c} M_r^r M_s^s \right] \tag{62}$$

and contains the operators

$$q_{r_1} \cdots q_{r_{N_c}}, \quad \tilde{q}^{s_1} \cdots \tilde{q}^{s_{N_c}}, \quad q_r \tilde{q}^s, \quad M_s^r . \tag{63}$$

Note the mixing between the singlets M_s^r and $q_r \tilde{q}^s$ through the infrared equations of motion:

$$q_r \tilde{q}^s = -\frac{m_0^*}{\lambda^*} \left[M_r^s - \frac{1}{N_c} \delta_r^s M_u^u \right] \tag{64}$$

The operator map (60) takes $Q^r \tilde{Q}_s$ to $q_s \tilde{q}^r$ and thus to M_s^r. The massless particles in the two theories are $Q^r, \tilde{Q}_s$ and $q_s, \tilde{q}^r$; one may verify that the 't Hooft anomaly matching conditions are trivially satisfied.

What happens in the limit $m_0 \to \infty$? We argued earlier that this theory should be dual to $m_0^* = 0$ simply from the enhanced flavor symmetries. The superpotential of the SQCMD theory is zero while that of the SQCMD* theory is $W = \lambda^* M_s^r q_s \tilde{q}^r$. The operator $q_r \tilde{q}^s$ is now redundant and disappears from the spectrum. The operator map (60) becomes, using (64),

$$Q^{r_1} \cdots Q^{r_{N_c}} \quad \leftrightarrow \quad \epsilon^{r_1 \cdots r_{N_c} s_1 \cdots s_{N_c}} q_{s_1} \cdots q_{s_{N_c}}$$

$$\tilde{Q}_{s_1} \cdots \tilde{Q}_{s_{N_c}} \quad \leftrightarrow \quad \epsilon_{s_1 \cdots s_{N_c} r_1 \cdots r_{N_c}} \tilde{q}^{r_1} \cdots \tilde{q}^{r_{N_c}} \tag{65}$$

$$Q^r \tilde{Q}_s \quad \leftrightarrow \quad M_s^r$$

The massless particles of the theories are $Q^r, \tilde{Q}_s$ in SQCMD and $q^r, \tilde{q}_s, M_s^r$ in SQCMD*. One may confirm [8] that these particles satisfy the 't Hooft anomaly matching conditions under the now enhanced $SU(2N_c)_L \times SU(2N_c)_R \times U(1)_B \times U(1)_R$ global symmetry.

The description of the previous paragraph coincides precisely with the N=1 duality map introduced by Seiberg [8] for the case $N_f = 2N_c$. A diagram illustrating our arguments appears in Fig. 7.

D. Other aspects of N=1 duality

We have identified a possible source for N=1 duality in the $SU(N_c)$ theory studied by Seiberg [8] with $N_f = 2N_c$. To a certain degree this is sufficient for $N_f \neq 2N_c$ as well. One may move from the known theory to any other by perturbing it by relevant operators, and, knowing how operators are mapped under N=1 duality at the initial point, we know how any given perturbation acts both in the original and in the dual theory. However, the full duality also involves non-perturbative effects [8] which our methods have not identified. What we may say with confidence is that the mere uniqueness of the anomaly-free R charge following perturbation by a relevant operator automatically guarantees the correct mapping of operators at a new fixed point and the matching of global anomalies found in [8]. This can be seen by considering infinitesimal mass and symmetry breaking perturbations, without integrating out any massive fields. The new flavor symmetries, and therefore the new flavor

anomalies, are linear combinations of the old ones; the linear combinations in the original theory are the same as those in the dual theory, so the new anomalies match. Those fields which are massive when the perturbations are finite have cancelling anomalies, of course, so when they are integrated out the anomalies still match. Similar arguments can be used for the operators. These arguments apply when both the original and dual theory flow to an interacting fixed point, which includes the cases $\frac{3}{2}N_c < N_f < 3N_c$.

For larger N_f the one-loop β-function is positive and the infrared fixed point must be free, while for smaller N_f there is a breakdown of unitarity [8] which implies that the description in terms of the original fields must fail. This and other subtleties (strong coupling, non-perturbative effects, confinement) are important for understanding the properties of and sometimes even the existence of certain fixed points. Insight into these issues might be gained by studying N=2 theories more thoroughly.

E. Commentary

We have presented a mechanism by which the S-duality of finite N=2 models could survive perturbation by an N=2 breaking term and could be transferred to theories which preserve only N=1 supersymmetry. We did so by giving a mass to the adjoint chiral superfield and showing that the resulting N=2̸ model had a marginal operator whose associated fixed curve was identical to that of another model, SQCMD. We noted that the endpoints of the curve ($m_0 = \infty, 0$) had enhanced flavor symmetries, and we used this fact to argue that the weak and strong coupling limits of the N=2 theory flowed to these two endpoints. By showing that the two endpoints were distinct theories, we found ourselves forced to conclude that S-duality could not map the N=2 theory onto itself. However, the large flavor symmetries of the N=2 theory strongly suggested that S-duality maps it to an N=2 theory with the same color and flavor representations. Since the mapping $\tau \to -1/\tau$ is a $\mathbb{Z}_2$ transformation, we were left with only one option; the duality transformation must act as a $\mathbb{Z}_2$ on the flavor space. Fortunately, there was a natural $\mathbb{Z}_2$ of the flavor $U(2N_c)$ — charge conjugation — which acted on operators in a simple way. When we brought our conjecture for the S-duality transformation of the N=2 theory into the N=2̸ model, and thereby into the SQCMD model, we found that the fixed curves of the SQCMD model and the dual SQCMD* model were mapped to one another, with large and small meson mass regions exchanging places. In particular the zero mass and infinite mass endpoints were mapped to one another. Analysis of the operator identifications, the massless fields, and the superpotentials showed that the mapping of the endpoints was precisely the N=1 duality mapping found by Seiberg [8] in the particular case $N_f = 2N_c$.

There are a number of gaps in our reasoning. We have assumed that the structure of the fixed curves is fairly straightforward and that there is no obstruction to traveling from $m_0 = \infty$ to $m_0 = 0$ in the SQCMD model. We did not prove that the strong coupling limit $\tau = 0$ of the N=2 model flows to the $m_0 = 0$ endpoint of the SQCMD model, which is critical for the argument. We also did not prove, much less derive, that the N=2 model is dual to itself up to a flavor transformation. Our choice for the $\mathbb{Z}_2$ flavor transformation appears to be unique, but perhaps there is a more subtle option. We have also neglected other aspects of N=2 duality, such as the existence of dyons. Insight into this and other related issues is to be found in [9]. We are optimistic that a substantial part of our mechanism is correct,

given the consistency of our picture with the various known examples [8,9,20].

Assuming that we have correctly identified the physical mechanism underlying N=1 duality, there are a number of comments to be made. First, the picture apparently extends to $SO(N_c)$ and $Sp(2N_c)$ models with vector multiplets; certainly most of the arguments go through. In the case of $Sp(2N_c)$ there are no new issues. However, for $SO(N_c)$, the flavor symmetry is $Sp(2N_c - 4)$, and a $\mathbb{Z}_2$ reflection in this space would not transform baryons (N_c-index antisymmetric tensors) into baryons in the dual theory. A more intricate operator mapping than Eq. (60) will therefore be necessary. This is presumably related to the complexity of N=1 duality for $SO(N_c)$ as compared with $SU(N_c)$. [8]

It is natural to try to generalize our mechanism to finite N=2 theories with more complicated representations; however, there are some obstacles. When superfields are integrated out of other N=2 models, the Fierz transformation used in (52) leads to a sum of a terms, not all of which can be written as a product of bilinear invariants. The introduction of a singlet auxiliary meson superfield then leaves other non-renormalizable operators in the superpotential whose coupling constant becomes large in the limit that the meson mass becomes small. Another problem is that the duality of N=2 models is even less well understood for other representations. For example, the gauge group of the dual theory is not necessarily the same as that of the original; indeed this is expected to occur in N=4 models.

If it is true that all N=1 duality stems exclusively from N=2 duality, the situation appears unfortunate for chiral models, which cannot be derived from N=2 theories. However, the finiteness of N=4 and N=2 models generalized to both vector-like and chiral N=1 models. Also, the E_6 model of Sec. IV F bore a close resemblance to the $SU(3)$ model of Sec. IV A, which had a duality transformation (to the model of Sec. VI C) on its fixed curve. This leads us to wonder whether even chiral models with fixed curves might have some form of weak-strong coupling duality.

One may also speculate as to whether the mechanism we have identified can be generalized in some way to bring N=1 duality down to non-supersymmetric theories. One might consider integrating out the gluinos, for example. We hope to return to this question in the future.

IX. CONCLUSION

We have shown that exactly marginal operators and extended manifolds of fixed points are commonplace in N=1 supersymmetric gauge theory. Our approach treats the previously known finite gauge theories and the specific case of $SO(4)$ studied in [8,9] on an equal footing. Many new examples were presented. Our exploration of models has not been by any means exhaustive, but we have found a wide range of interesting theories which can be expected after further study to reveal a variety of new phenomena. We saw that manifolds of fixed points may be weakly or strongly coupled; they may be generated by renormalizable or non-renormalizable operators; theories containing them may be chiral or vector-like. Our examples have included finite models, models with curves of fixed points which are everywhere weakly coupled but nowhere free, models with strongly coupled manifolds of fixed points, and models which have fixed curves which run to infinite gauge coupling. The limitations of our methods are still unclear, but the consistency of the picture we advocate here is strong evidence that they hold in many interesting cases.

The issue of duality has come up in several contexts. We have pointed out several finite N=1 models that may have S-duality of some form. The N=1 duality of Seiberg [8,9] has been essential in giving us confidence that manifolds of fixed points are really present in non-renormalizable models. We have also identified many models in which the fixed curves of finite models flow to or from fixed curves of theories that are perturbatively non-renormalizable; those which stem from finite N=2 models may well have a form of N=1 duality.

Of these, the models studied in [8] with gauge group $SU(N_c)$ and $N_f = 2N_c$ flavors are of great interest. It was suggested in [8] that the N=1 duality studied therein might be related to duality of the finite N=2 model with $N_f = 2N_c$. We have suggested a physical mechanism by which this relationship could be realized. The electric and magnetic descriptions of the N=2 fixed curve flow under an N=1 breaking perturbation to dual descriptions of an N=1 fixed curve. With a few reasonable assumptions, we have demonstrated that one point of this curve is described by the electric and magnetic theories of [8] with $N_f = 2N_c$. This certainly does not constitute a derivation of N=1 duality from N=2 duality; our arguments are speculative. Still, their consistency with Refs. [8] and [20] makes us optimistic that they will prove, at least in part, to be correct.

The exploration of the space of four dimensional conformal field theories has just begun. We hope that, as in two dimensions, the study of marginal operators will open many new paths for investigation.

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FIGURES

FIG. 1. The renormalization group flow near the N=4 fixed line, shown schematically, in the plane of the gauge coupling g and the Yukawa coupling h. Arrows indicate flow toward the infrared.

FIG. 2. The renormalization group flow near the fixed curve of the model of section Sec. V A, shown schematically, in the plane of the two gauge couplings. Arrows indicate flow toward the infrared.

FIG. 3. The renormalization group flow near the fixed curve associated with the superpotential (28), shown schematically, in the plane of the gauge coupling g and the Yukawa coupling h. Arrows indicate flow toward the infrared.

FIG. 4. The renormalization group flow near the fixed curve of the model of section Sec. VI C, shown schematically, in the plane of the gauge coupling g and the Yukawa coupling h. Arrows indicate flow toward the infrared.

FIG. 5. The SQCMD theory flows to the same fixed curve as the N=$\not{2}$ model (a finite N=2 model broken by a mass term $m_\Sigma \Sigma^2$ for the adjoint chiral superfield.) Along the fixed curve the flavor symmetry is $U(N_c)$, except at the endpoints where it is $SU(N_c) \times SU(N_c) \times U(1)$. The theory at $h = 0$ ($m_0 = \infty, \tau \to i\infty$) is different from that at $h = \infty$ ($m_0 = 0 = \tau$).

FIG. 6. The duality of N=2 maps the electric theory with τ to the magnetic N=2* theory with $\tau^* = -1/\tau$. The duality is carried down to the infrared fixed curves of the N=$\not{2}$ and N=$\not{2}^*$ models. Note that the endpoints of the fixed curves are mapped to one another.

FIG. 7. The duality of the N=$\not{2}$ and N=$\not{2}^*$ models translates into an infrared duality for the SQCMD and SQCMD* models. The theory at $m_0 = \infty$ is equivalent under duality to the theory at $m_0^* = 0$; these limits of SQCMD and SQCMD* were first identified as dual by Seiberg [8].

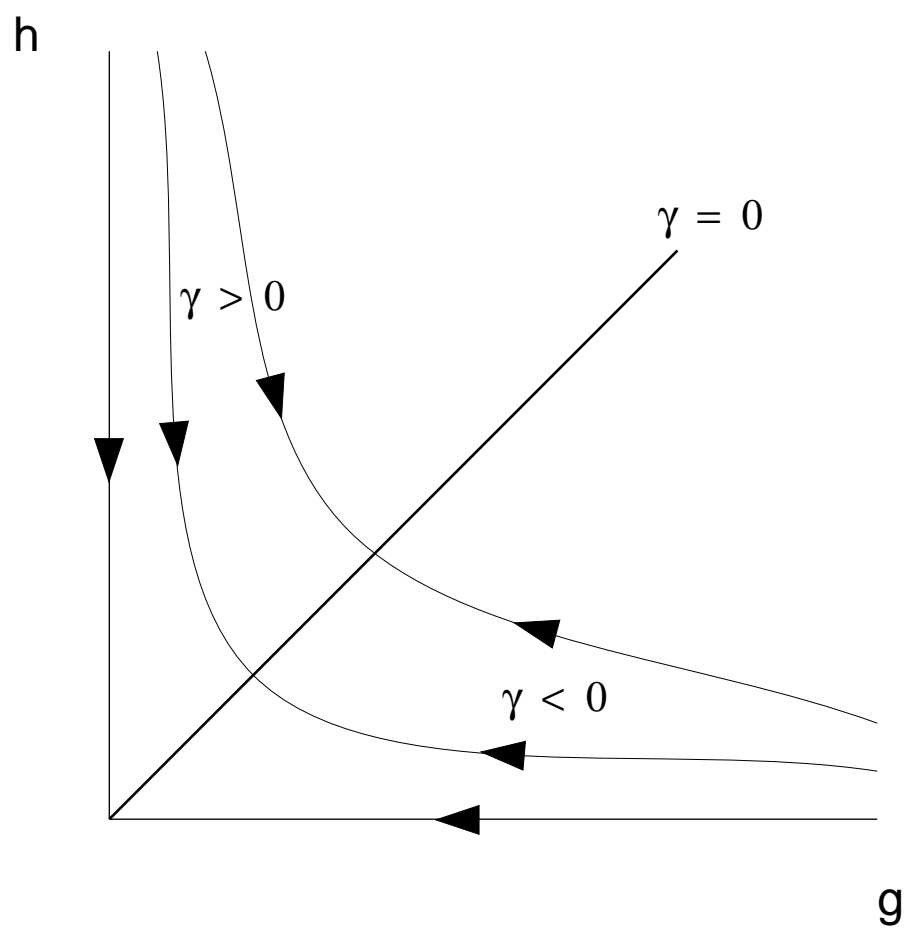This figure "fig1-1.png" is available in "png" format from:

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Figure 1

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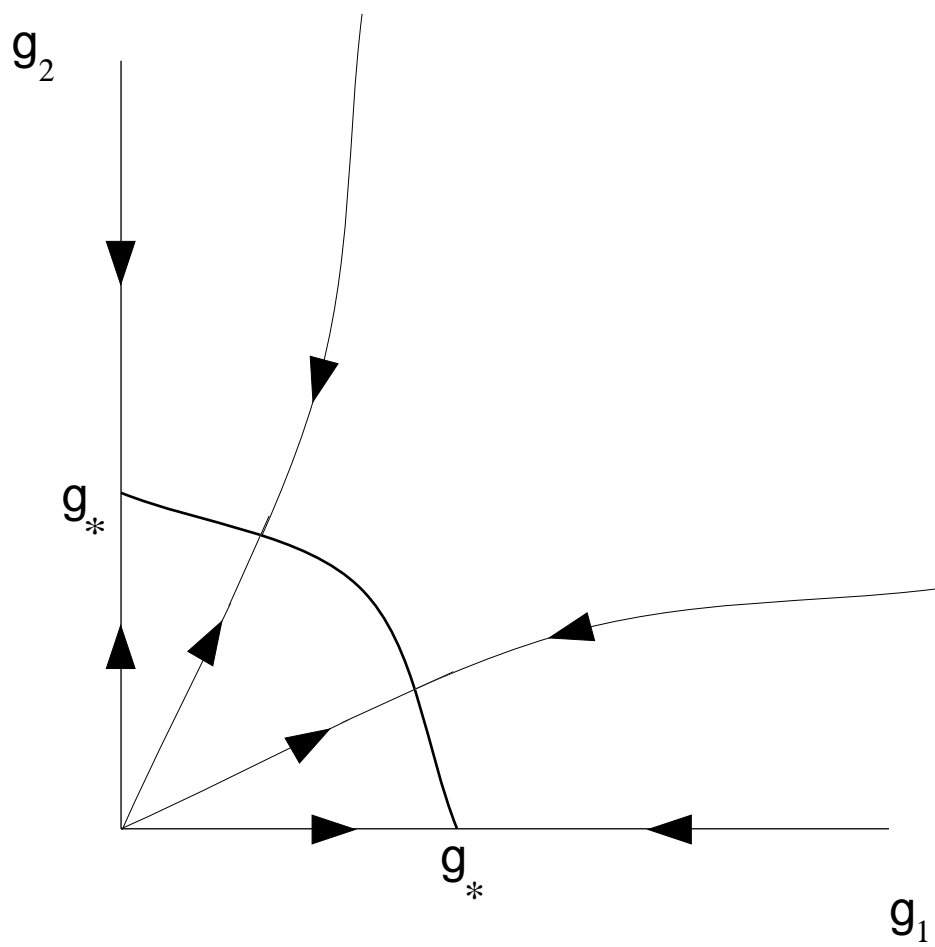
Figure 2

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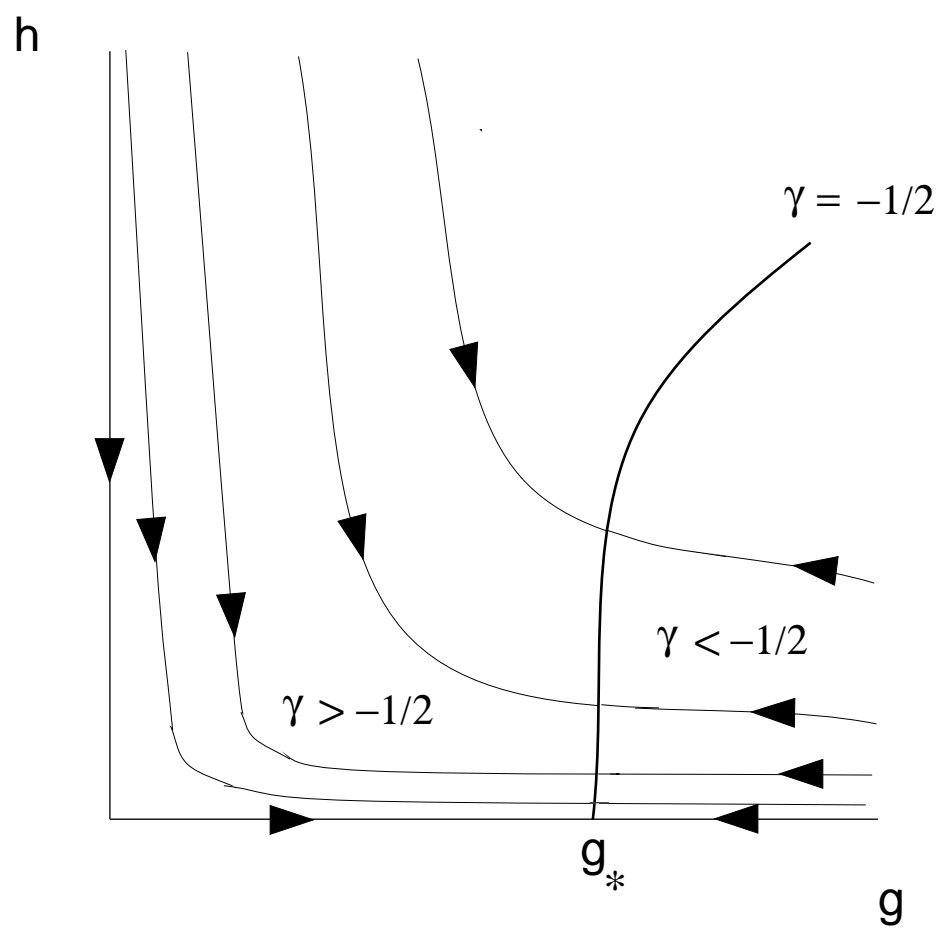
Figure 3

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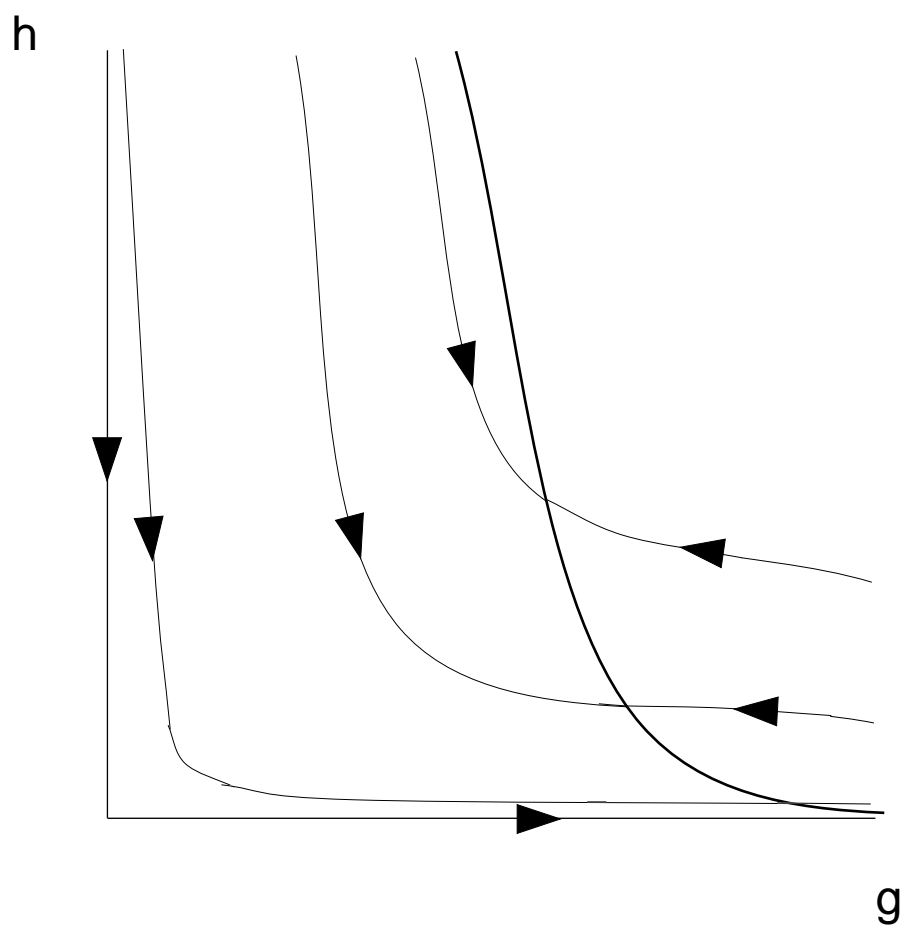
Figure 4

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SQCMD   N = 2

$m_0 = 0$ $\tau = 0$

$m_0 = \infty$ $\tau \longrightarrow i\infty$

$h = \infty$ m_Σ

                                          N = 2

$h = 0$

SU(N$_c$) with
2N$_c$ flavors
$Q^r, \ \tilde{Q}_s$
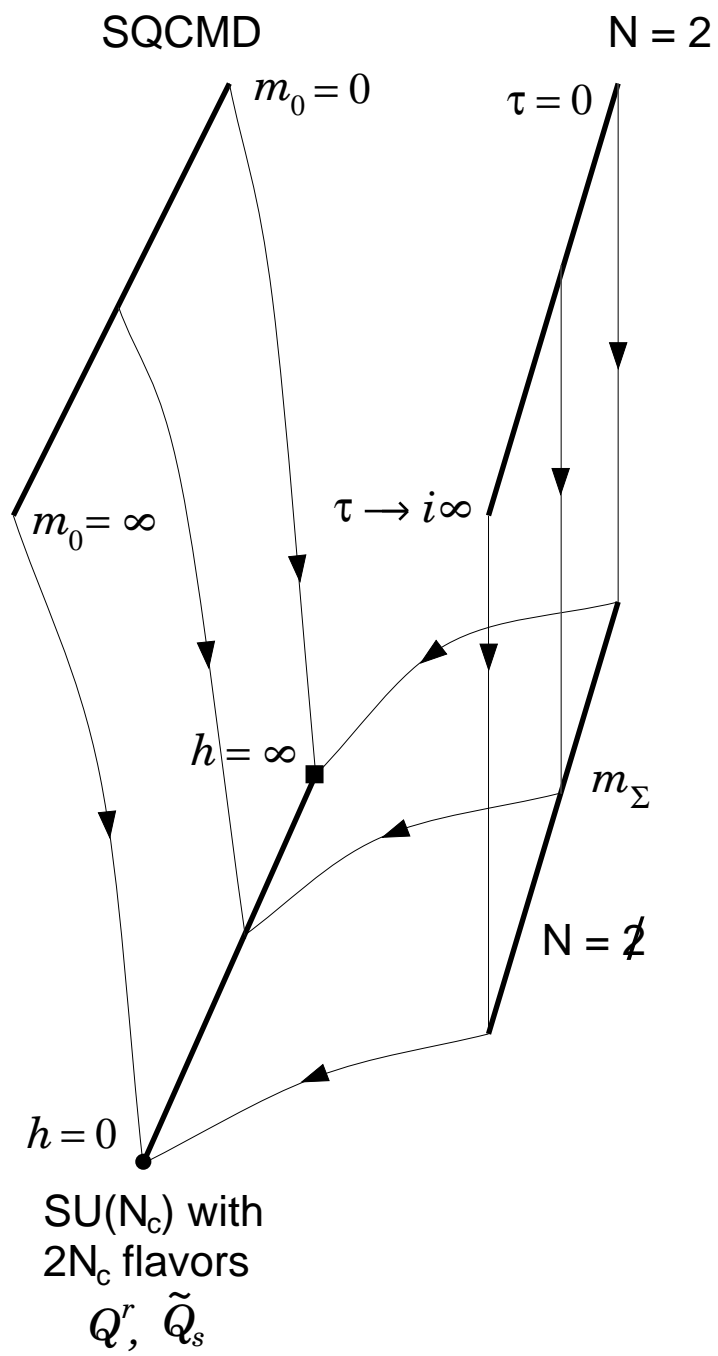
Figure 5

This figure "fig1-6.png" is available in "png" format from:

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N = 2   N = 2*

$\tau = 0$ $\tau* \longrightarrow i\infty$

$\tau \longrightarrow i\infty$ $\tau* = 0$

SU(N_c) with
2N_c flavors
q_r, $\tilde{q}^s$

$h* = 0$

N = 2   N = 2*

m_Σ

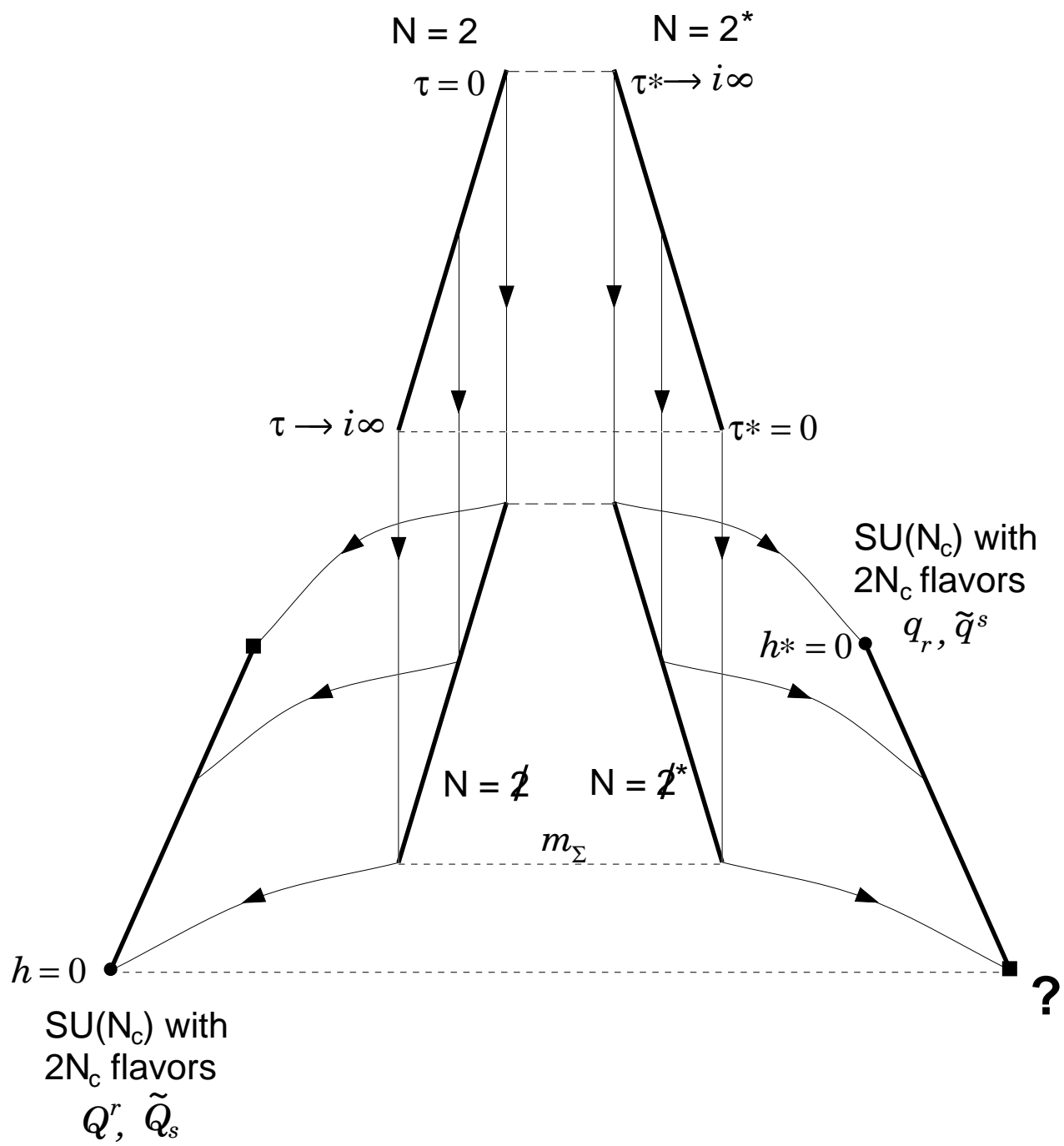$h = 0$

SU(N_c) with
2N_c flavors
Q^r, $\tilde{Q}_s$

?

Figure 6

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SQCMD $\quad$ $m_0 = 0$ $\qquad$ N = 2 $\qquad$ N = 2* $\qquad$ $m_0^* = \infty$ $\quad$ SQCMD*

$\tau = 0$ $\qquad$ $\tau* \rightarrow i\infty$

$m_0 = \infty$ $\qquad$ $\tau \rightarrow i\infty$ $\qquad$ $\tau* = 0$ $\qquad$ $m_0^* = 0$

$h* = 0$

N = 2 $\qquad$ N = 2*

m_Σ

$h = 0$

SU(N$_c$) with
2N$_c$ flavors
$Q^r, \ \tilde{Q}_s$

SU(N$_c$) with
2N$_c$ flavors
and singlet mesons
$q_r, \ \tilde{q}^s, \ M_s^r$

$W = 0$ $\qquad\qquad$ $W = M_s^r q_r \tilde{q}^s$

Figure 7